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Chiral corrections in electroweak processes with heavy mesons

Ph.D. Thesis

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Ljubljana, 2002

TO ANDREJA

Many thanks go to the people at the Department of Theoretical Physics at Jožef Stefan Institute. For many enlightening discussions I am especially indebted to Borut Bajc, Damjan Janc, Matjaž Poljšak, Saša Prelovšek Komelj, and of course to my advisor Svjetlana Fajfer, that has managed to guide me through Scylla and Charybdis with many invaluable suggestions. I would also like to thank Damir Becirevic, Jan Olav Eeg, Yuval Grossman, Sourov Roy, and Paul Singer, for widening my horizons with many ideas and insights, as well as for successful collaboration.

It is difficult to thank enough to those, that I hold dear, Andreja, my parents, and friends, who have had enough patience with me over the long hours spent behind the computer screen in preparation of this text.

I would like to acknowledge that this work was supported in part by the Ministry of Education, Science and Sport of the Republic of Slovenia. I am also grateful to the High Energy Physics Group at the Israel Institute of Technology-Technion for the hospitality during winter 2002, where part of this work has been done.

Abstract

The effective theory based on combined chiral and heavy quark symmetry, the heavy hadron chiral perturbation theory, is applied to D meson decays. In $D^0 \to K^0 \bar{K}^0$ decay the nonfactorizable contributions are calculated. These arise from chiral loops and products of color-octet currents, while the prediction vanishes in the factorization limit. The approach is confronted with the experimental data. Next, the flavor changing neutral current rare charm decays are considered. The predictions for $c \to u l^+ l^-$, $D^0 \to \gamma\gamma$, and $D^0 \to l^+ l^- \gamma$ are given both in the Standard Model as well as for the Minimal Supersymmetric Standard Model with and without R parity conservation. A possible enhancement of order 50 compared to the Standard model prediction is found for the $D^0 \to \mu^+ \mu^- \gamma$ channel. This makes it an interesting probe of New Physics.

A modified version of the heavy hadron chiral perturbation theory is used to estimate effects of quenched approximation in the lattice calculations of $B \to \pi, K$ transitions. The relevant form factors, $F_{+,0}$, contain the chiral quenched logarithms that diverge in the chiral limit $m_\pi \to 0$. Behavior of the form factors as functions of m_π^2 in quenched and full QCD is then found to be substantially different in the region close to the physical pion mass.

In the thesis several technical details are clarified as well. The explicit calculation of three and four-point scalar functions with one heavy-quark propagator is given. Next, existing renormalization group evolutions for B and K meson decays are modified to perform next-to-leading order evolution of Wilson coefficients for charm decays. Also a discussion of gauge invariance in effective theories is given.

`Key Words:` flavor changing neutral current, weak decays of heavy mesons, heavy meson chiral perturbation theory, rare radiative decays, new physics searches, quenched approximation, lattice quantum chromodynamics

`PACS:` 13.25.Ft, 13.20.-v, 13.60.-r, 12.60.Jv, 12.38.Gc

Notation

The characters from the middle of the Greek alphabet $\mu, \nu, \ldots$ in general run over space-time indices $0, 1, 2, 3$, while the Latin indices $i, j, k, \ldots$ run over spatial indices $1, 2, 3$.

The metric used in the thesis is $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, where the indices μ, ν run over $0, 1, 2, 3$, with 0 the temporal index.

The Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ is defined as a totally antisymmetric tensor with $\epsilon^{0123} = 1$.

The Einstein summation over repeated indices is assumed unless stated otherwise. The dot-product $p{\cdot}k$ denotes $p^\mu k_\mu$.

The Dirac matrices are defined so that $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$. Also, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The left and right-chirality projection operators are $P_L = \frac{1}{2}(1 - \gamma_5)$ and $P_R = \frac{1}{2}(1 + \gamma_5)$. The matrix $\sigma^{\mu\nu}$ is $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The slash on a character denotes $\not{p} = p^\mu\gamma_\mu$.

The trace Tr runs over the Dirac indices, while the lower case trace tr runs over the $SU(3)$ flavor indices.

The complex conjugate and Hermitian adjoint of a vector or a matrix A are denoted A^* and $A^\dagger$ respectively. A hermitian adjoint of an operator O is denoted $O^\dagger$. A bar on a Dirac bispinor u denotes $\bar{u} = u^\dagger\gamma_0$.

The imaginary and real part of a complex number z are denoted $\Im(z)$ and $\Re(z)$ respectively.

The Heaviside function $\Theta(u)$ is defined as $\Theta(u) = 1$ for $u > 0$ and zero otherwise.

Natural units with $\hbar$ and the speed of light taken to be unity are used. The fine structure constant is thus $\alpha_{\mathrm{QED}} = e^2/4\pi \simeq 1/137$.

Chapter 1

Introduction

Particle physics has gone a long way from its beginnings in the first half of the 20^{th} century. From the present perspective it is actually hard to imagine, what the world was like without the "Standard Model" of elementary particle physics[1]. The gauge-field theoretical description of fundamental electromagnetic, weak, and strong interactions, that emerged in the 1960's, has completely dominated the field ever since.

The structure of the Standard Model is as follows. Its building blocks are fermions, leptons and quarks [**?**], that come in three families. The Standard Model gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$, where the $SU(3)_c$ is the gauge group of Quantum Chromodynamics (QCD) [**?**], $SU(2)_L$ is the gauge group of weak isospin, while $U(1)_Y$ is the gauge group of weak hypercharge [**?**]. The classification of the leptons and quarks appearing in the Standard Model according to the weak isospin is

$$\begin{array}{llll} \text{leptons} & & & \\ & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L & \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \\ & e_R & \mu_R & \tau_R \\ \text{quarks} & & & \\ & \begin{pmatrix} u \\ d' \end{pmatrix}_L & \begin{pmatrix} c \\ s' \end{pmatrix}_L & \begin{pmatrix} t \\ b' \end{pmatrix}_L \\ & u_R & c_R & t_R \\ & d'_R & s'_R & b'_R \end{array}$$

where the binomials with the subscript L denote the weak isospin doublets. Leptons are color singlets, while quarks are in the fundamental representation of $SU(3)_c$. The masses of leptons and quarks are generated via Higgs mechanism [**?**]. This also gives masses to the $W^\pm$ and Z bosons and breaks the electroweak gauge group $SU(2)_L \times U(1)_Y$ to the electromagnetic $U(1)$. Because there are no right-handed partners of the left-handed neutrinos, these are left massless in the Standard Model (SM), if only renormalizable terms are present.

It is customary to use the mass eigenbasis instead of the weak basis for the quark fields. The rotation to the mass eigenbasis is conventionally conveyed to the down-quark fields

The successes of the Standard Model (SM) description are abundant. To name just the recent few: electroweak precision tests are generally in impressive agreement with the SM predictions

[1] The name was apparently bestowed by Sam B. Treiman [**?**].

[?, ?], the CP violation experiments in B and K meson systems support the CKM description of the Standard Model with one universal phase [?, ?, ?], the discovery of t-quark in the mass range predicted from the electroweak precision data was a triumph of the SM [?, ?]. All in all, there is just one missing building block, the discovery of Higgs boson, that would make the picture complete. The direct searches at LEP give the current lower limit $m_H > 114.4$ GeV at the 95% confidence level [?, ?]. The indirect experimental constraints are obtained from the precision measurements of the electroweak parameters, which depend logarithmically on the Higgs boson mass through radiative corrections. Currently these measurements constrain the Standard Model Higgs boson mass to $m_H = 81^{+51}_{-33}$ GeV or to values smaller than 193 GeV at the 95% confidence level [?].

From a theoretical point of view, the Standard Model has also quite a few very attractive features. First of all, it is a renormalizable theory. This means that it is very predictive. Using a relatively small set of parameters, masses of quarks and leptons, masses of gauge bosons and the values of coupling constants, all in all of order 20 [2], one is able, at least in principle, to predict a myriad of processes. Because of renormalizability no additional infinite terms are generated by quantum effects, so that in principle the validity of the SM can be extended to arbitrary high scales.

However, we know from the observations, that the SM cannot be the end of story. First of all, gravity is not included in the Standard Model. The quantized description of gravity has proved to be a very challenging subject, that has kept theorists busy for the past two decades with an especially extensive work done in the field of string theories [?]. No experimental insight is available in this area, though. Next, recent data from Superkamiokande [?] and SNO [?, ?] have provided a solid experimental evidence for neutrino oscillations. These imply nonzero neutrino masses, contrary to the SM description. Another phenomenological indication of non-Standard Model physics is the unification of strong, electromagnetic and weak couplings in the context of supersymmetric grand unified theories (SUSY GUTs) at the scales of $\mathcal{O}(10^{16}$ GeV) [?, ?]. Very solid experimental data suggesting non-SM physics are coming from astrophysics and cosmology. The astrophysical observations suggest that most of the matter in the Universe is not luminous, but dark [?]. Most of the dark matter also is not baryonic. The nonbaryonic dark matter can either be cold or hot, but the general consensus is that most of it must be cold. The Standard Model does not provide a candidate for nonbaryonic cold dark matter, while for instance a very appealing candidate is provided by the lowest supersymmetric candidate, the neutralino. Another evidence pointing toward SM extensions is the generation of baryon-antibaryon asymmetry in the early Universe. To generate it, the interactions between particles should be CP violating. The CP violation is present in the SM, but is not strong enough to account for the observed asymmetry [?, ?].

There are also some conceptual problems with the structure of the Standard Model. The running of coupling constants suggests the unification scale at $10^{14} - 10^{16}$ GeV.[3] In view of this large scale, the weak scale $1/\sqrt{G_F} \sim 250$ GeV suddenly appears to be very small. The large difference between the two scales cannot be explained "naturally" in the context of the SM. This " hierarchy problem" is connected to the fact that the theory contains a fundamental scalar field, which receives quadratically divergent loop contributions to the mass parameter. Taking the cutoff in regularization prescription to represent the scale of new physics, the values of the bare mass and the loop corrections to it have to be fine-tuned to give the small physical mass. This is the case,

[2] More precisely 3 lepton masses, 6 quark masses, 4 CKM parameters, 3 gauge coupling constants, mass of the Higgs boson and the quartic coupling λ give altogether 18 parameters. Counting in also the strong CP parameter θ this amounts to 19 parameters of the renormalizable Standard Model.

[3] Note that precise unification of couplings does not occur with running of coupling constants given only by the Standard Model fields [?].

unless the scale of new physics is close to the weak scale. The hierarchy problem can be solved in several different ways. If the fundamental Higsses exist, the theory can be stabilized by TeV scale supersymmetry [?, ?]. The other option is that Higgs is a composite object, that is either a bound state of fermions or a condensate. Technicolor theories represent a class of proposals along the latter lines [?]. Another solution to the hierarchy problem has been suggested recently [?, ?]. If additionally to the usual 3+1 space-time dimensions, "large" compact dimensions are assumed, the scale of gravity is much lower than the Planck scale. For two sub-mm extra dimensions the scale of gravity is in the TeV range.

Another challenging conceptual problem is coming from the cosmological observations of distant supernovae type Ia explosions, that suggest a nonzero cosmological constant [?, ?]. The corresponding energy density is of the order of present critical density of the Universe $\rho_c \sim 10^{-26}$kg m^{-3} $\sim 10^{-14}$ eV^4. If this is to be explained by the vacuum expectation values and the chiral condensates of the SM fields that correspond to energy scales from a few 100 MeV to a few 100 GeV, one would need an incredibly fine-tuned cancellation between various contributions to arrive at the correct value of the cosmological constant. Note, that a number of alternative explanations for the dimming of supernovae have also been proposed [?, ?, ?, ?], some of them requiring new physics beyond the SM.

Given the discussion above, the modern point of view is to consider the Standard Model "merely" as an effective field theory. In the effective field theory description one usually has two scales with a very distinct hierarchy and the intermediate scale μ, that separates the two. The physics at the lower scale can then be described by means of a Wilsonian expansion $\mathcal{L} \sim C_i(\mu)Q_i(\mu)$, where the higher scale physics is hidden in the coefficients $C_i(\mu)$, while operators $Q_i(\mu)$ incorporate the lower scale physics. In the Standard Model only the renormalizable operators appear. Operators of higher dimensions are suppressed by the high scale, e.g., by the GUT scale, and can break the conservation laws of the SM only weakly.

In the effective field theories, one can distinguish between two approaches, the "bottom-up" or the "top-down" approach. In the "top-down" approach, the high-scale physics is well understood and the coefficients C_i are calculable, for instance in the perturbative framework. The prominent example of this approach is the operator product expansion applied to the weak decays [?]. In this case the "high scale theory", the electroweak theory, is well understood. For processes at energies $E \ll \mu_W \sim 90$ GeV, the W and Z bosons can be integrated out. In this way one effectively gets the old Fermi theory of weak interactions, but with calculable corrections to $(V-A)(V-A)$ contact interaction. Viewing the Standard Model as an effective theory, on the other hand, represents the "bottom-up" approach, where little is known about the high energy physics.

An example of the "bottom-up" approach is also the application of the effective theory concepts to strong interactions. QCD is a well understood theory, however, the low energy processes are in the nonperturbative region, where an expansion in the coupling constant is no longer applicable. Calculations ab initio, i.e., by starting with the QCD Lagrangian and finishing up with the predictions for physical observables, are still possible through the use of lattice QCD techniques, but are computationally very challenging [?]. Lattice methods also have their own limitations. To get meaningful results, calculations have to be done in Euclidean space-time, which makes the calculations of decay processes with more than one hadron in the final state very hard. Also, in order to make the numerical difficulties tractable, a number of approximations have to be made, e.g., by neglecting sea-quark effects, or by working at relatively high pion masses. Another option, that has been commonly used in the past, is to use the symmetries of the QCD Lagrangian to construct effective theories. Unknown couplings in the effective theory are then fixed from experiment. If such an effective theory contains a small expansion parameter, it can be predictable, with more experimental processes predicted than there are parameters to be fixed from experiments. The small expansion parameter for the chiral perturbation theory (χPT) is provided by the small momenta of interacting Goldstone bosons and by the small masses of u, d, s quarks, with $m_s \sim 100$ MeV still significantly smaller than the chiral scale ~ 1 GeV [?, ?, ?, ?]. A different approximate symmetry is used to construct the heavy quark effective theory (HQET) [?, ?, ?]. This is obtained when masses of b and c quarks are taken to be very large. Both chiral and heavy-quark symmetries can be combined in processes involving single heavy hadron, resulting in a heavy hadron chiral perturbation theory (HHχPT) [?].

In this thesis several applications of the effective theory concepts will be made [?, ?, ?, ?, ?, ?, ?]. The main focus will be on the application of heavy hadron chiral perturbation theory to D meson decays. We use the leading terms in the $1/m_c$ expansion and the expansion of momenta. Note, that in principle also higher order terms in the expansion could be important. Keeping only the leading order terms in the expansion has, however, several important advantages as (i) the set of unknown parameters is relatively small, (ii) there are enough experimental data to fix all of them, (iii) gauge invariance at 1-loop in HHχPT is obvious. Neglecting higher order terms can then be also viewed as a part of our model. The idea has been tested on the example of $D^0 \to K^0\overline{K^0}$, where good agreement with experiment has been found [?]. To appreciate this fact, one has to keep in mind that the commonly used factorization approximation predicts vanishing branching ratio for this decay mode, in disagreement with experimental data.

The same theoretical framework is then applied to the rare $D^0 \to \gamma\gamma$, $D^0 \to l^+l^-\gamma$ decays [?, ?, ?, ?]. The area of rare heavy meson decays has received a boost with the onset of B-factories. One of the goals of Belle and BaBar has been to pinpoint the CP violating mechanism and to further constrain the CKM matrix elements. But a considerable part of experimental efforts constitute the searches for rare decays [?]. Rare decays are especially interesting, if they are connected to a conservation law. Several such selection rules are present in the SM, with $\mu \to e\gamma$ and proton decay for instance completely forbidden at perturbative level in the renormalizable SM, while they can occur in scenarios beyond the SM. But also processes that are not completely forbidden in the Standard Model can be extremely useful as probes of new physics. For instance, the flavor changing neutral currents (FCNCs), i.e., transitions of type $q_i \to q_j + \gamma(\nu\bar{\nu}, l^+l^-)$, do not occur in the SM at tree level. They do, however, occur at the loop level, but are suppressed because of the Glashow-Iliopoulos-Maiani (GIM) mechanism and because of the hierarchical structure of the CKM matrix elements. The FCNCs can be significantly affected by the possible new physics effects, that either contribute at tree level or in the loops. Note that new physics could affect the FCNCs in a substantially different manner than the $\Delta F = 2$ and the charged currents, from which the constraints on CKM matrix elements are obtained at present.

Phenomenologically very exciting are the so-called golden-plated modes. For the decay mode to be golden-plated, it has to fulfill several requirements: (i) the SM amplitude has to be either very small or completely forbidden, (ii) it has to be theoretically clean, with small or virtually no uncertainties due to the nonperturbative strong interaction physics, (iii) it has to receive potentially large contributions from new physics scenarios. Examples of such golden modes are $K \to \pi\nu\bar{\nu}$ and $B \to X_s\nu\bar{\nu}$, where theoretical uncertainties due to the nonperturbative physics are very small [?]. No such golden modes are present in the charm physics. In rare D decays the nonperturbative physics of light quarks is expected to dominate the decay rates. Consider for instance the case of $c \to u\gamma$ transition that occurs only at the one loop level in the Standard Model. The contributions coming from b, s, d quarks running in the loop are

Because LD effects dominate in D decays, no extraction or tests of CKM matrix are possible in these decays. Also, in order to be able to probe new physics, its effects, if present, have to be large. However, there is an important sidepoint to the whole story. Namely, D physics probes the flavor structure of up-quark sector, in contrast to K and B decays. The non-SM extensions of up and down-quark sectors can be very different. In this sense rare D meson decays can prove as a valuable probe of new physics effects. In this thesis possible effects of supersymmetric extensions of the SM to the $c \to ul^+l^-$, $D^0 \to \gamma\gamma$, $D^0 \to l^+l^-\gamma$ decays will be considered.

Finally, the use of effective theories can be made also in the lattice QCD ab-initio calculations [**?**]. This is not surprising, given that there are many different scales present in the problem, the masses of quarks m_Q, m_q, the nonperturbative scale Λ, as well as the UV and IR cutoff scales set by the lattice spacing a and the size of the lattice L. In the ideal case they exhibit a hierarchy

The outline of the thesis is as follows. In the first three chapters we introduce the prerequisites for the phenomenological studies in the subsequent chapters. In chapter 2 we introduce the concept of effective field theories with a focus on χPT and HQET. Gauge invariance is discussed as well. In chapter 3 we work out the technical details connected with the integration of two-, three- and four-point functions in HQET. Chapter 4 is devoted to operator product expansion and the application to weak interactions. The factorization approximation is discussed in the same chapter. In chapter 5 the theoretical framework is applied to $D^0 \to K^0\overline{K^0}$ decay, while in chapter **??** rare D decays $D^0 \to \gamma\gamma$ and $D^0 \to l^+l^-\gamma$ are estimated both in the SM and in the supersymmetric extensions. Finally, in chapter **??** quenching errors in $B \to \pi$ transitions are discussed. Further technicalities and explicit results of the calculations are relegated to the appendices.

Chapter 2

Heavy quark effective theory and chiral expansion

2.1 Effective theories

The persisting problem of the phenomenological calculations in hadronic physics is the nonperturbative nature of strong interactions. In the past three decades the approach of effective theories has proved to be an extremely important tool in these considerations. As is usual in contemporary physics, the hard problems are simplified or avoided entirely by the use of approximate and/or exact symmetries. As will be shown below, the use of symmetries is also the common feature of the effective theories.

First let us introduce the notion of the effective quantum action. The nonperturbative definition can be found e.g in chapter 16. of [?], while we will discuss only the perturbative definition of the effective action[1]. Let us consider a general quantum field theory with a set of fields ϕ^r. The observables of the theory are deduced from the appropriate Green's functions. Perturbative calculations of these consist of tree as well as of loop diagrams. The *effective quantum action* $\Gamma[\phi]$ is such an action, that reproduces the Green's functions of the original theory exactly, but that is used only at tree level in the perturbative expansion. In terms of path integrals

$$\int [d\phi]\phi^{r_1}\cdots\phi^{r_n}e^{i\int d^4x\mathcal{L}(\phi)} = \int_{\text{TREE}}[d\phi]\phi^{r_1}\cdots\phi^{r_n}e^{i\Gamma[\phi]}, \tag{2.1}$$

where $[d\phi]$ denotes the path integral measure and the integration on the right-hand-side is only over the tree diagrams. Pictorially, using the effective action $\Gamma[\phi]$ instead of the original action $\int d^4x\mathcal{L}(\phi)$ means that the parts of diagrams containing loops can be replaced with blobs representing effective vertices (see Fig. 2.1). Only tree level diagrams are then left in the calculation.

There is a very useful theorem connected with the effective action. It states that if the original action $I[\phi] = \int d^4x\mathcal{L}[\phi]$ and the integration measure are invariant under the *linear* infinitesimal transformations

$$\begin{aligned}\phi^r(x) \to \phi^r(x) &+ \epsilon\mathcal{F}(\phi^r;x),\\ \mathcal{F}(\phi^r;x) &= s^r(x) + \int t^r_{\bar r}(x,y)\phi^{\bar r}(y)d^4y,\end{aligned} \tag{2.2}$$

[1]The effective action has been first introduced perturbatively in [?], while a nonperturbative definition was first given in [?].

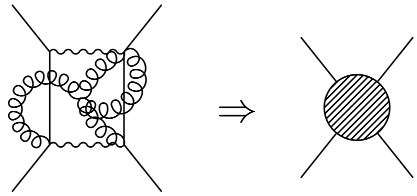


Figure 2.1: One particle irreducible parts of the diagram are replaced by effective vertices- the loops are replaced by the blob, i.e., the tree level effective vertex.

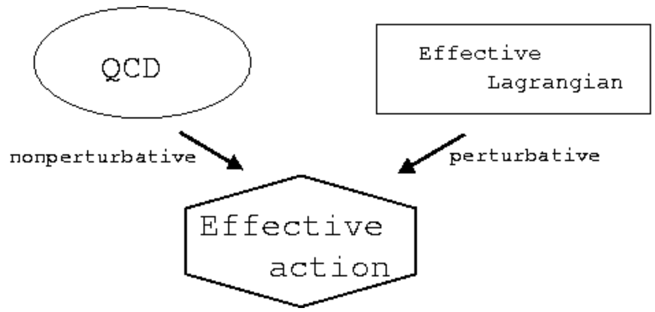


Figure 2.2: Both QCD and the effective Lagrangian corresponding to QCD lead to the same effective action. The connection between QCD and the effective action is nonperturbative, between the effective Lagrangian and the effective action on the other hand it is perturbative

where $s^r(x)$ and $t^r_{\bar{r}}(x,y)$ are c-number functions, then the effective action $\Gamma[\phi]$ is also invariant under the same transformations. Note that this is not necessarily true for nonlinear infinitesimal transformations, where effective action in general will not be invariant under the same transformations as the original action. For proof and further details see [**?**].

Now we turn to the notion of effective Lagrangian, with the main idea depicted on Fig. 2.2 for the case of QCD. Let us suppose that the initial Lagrangian $\mathcal{L}$ consists of a set of fields that transform linearly under a group G, and that the Lagrangian itself is invariant under G. For the case of QCD the fields will be the light quarks transforming under $SU(3)_L \times SU(3)_R$. Since the realization of G is linear, also the effective action $\Gamma[\phi]$ is invariant under G. The calculation leading from the initial Lagrangian $\mathcal{L}$ to the effective action $\Gamma[\phi]$ can be highly nontrivial, with a possible nonperturbative regime as is the case for the low-energy QCD. It is then useful to construct from the relevant fields ϕ (for the case of low-energy QCD, these are the pseudoscalar fields or any other fields relevant for the processes considered) the most general Lagrangian $\mathcal{L}_{\rm eff}(\phi)$ invariant under G. In general this will consist of an infinite number of terms with unknown couplings. The procedure will be useful if we find a rationale to keep just a finite number of terms. In the case of chiral perturbation theory this is provided by an expansion in momenta, while in heavy quark effective theory the expansion is in the inverse of the heavy-quark mass.

If the fields in the effective Lagrangian $\mathcal{L}_{\rm eff}$ transform linearly under G, also the effective action $\Gamma[\phi]$ following from the effective Lagrangian $\mathcal{L}_{\rm eff}$ will be invariant under G. We can then perform the perturbative calculation using the effective Lagrangian $\mathcal{L}_{\rm eff}$ to some fixed order and predict the effective action $\Gamma[\phi]$. At each order a number of unknown couplings have to be determined from the experiment. The number of couplings grows with the higher order contributions, so that the effective Lagrangian approach becomes less and less predictive when going to higher orders.

The method of effective Lagrangians has been very successfully applied to the case of spontaneously broken global symmetries. In particular it is very successful in the effective description of strong interactions in the low energy regime. We thus illustrate the procedure for this case. We start with the underlying “fundamental” Lagrangian $\mathcal{L}$. Let us suppose that the fields ϕ^r transform linearly under some continuous transformation group G

$$\phi' = g\phi, \tag{2.3}$$

where $g \in G$, while we have introduced the column ϕ containing all the fields. If the initial Lagrangian $\mathcal{L}(\phi)$ is invariant under G, so is the effective action $\Gamma[\phi]$. This is the case also if G is spontaneously broken down to some subgroup H. The important artifact of the spontaneously broken global symmetries are the massless Goldstone boson fields that parametrize the G/H right coset space. Since they are massless, it is useful to factor them out of the other fields appearing in the problem. We introduce

$$\phi(x) = \xi(\varphi(x))\tilde{\phi}(x), \tag{2.4}$$

where φ are the Goldstone boson fields. It is always possible to choose the function $\xi(\varphi(x))$ such that $\tilde{\phi}(x)$ do not contain Goldstone boson fields (for details see chapter 19 of [?], or [?, ?]). Since the subgroup H is unbroken, it is always possible to redefine $\xi(\varphi(x)) \to \xi(\varphi(x))h$, $h \in H$, without changing the effective action. As already stated, the $\xi(\varphi(x))$ is then a representative of the right coset, with $\xi(\varphi(x))h$ being equivalent for all $h \in H$. A very common parametrization of the right coset space G/H is

$$\xi(\varphi(x)) = e^{i\varphi^a(x)x_a}, \tag{2.5}$$

with x_a the generators of broken symmetries (the independent vectors in the Lie algebra of G that do not belong to the Lie algebra of H). The fields $\tilde{\phi}$, φ transform under G as

$$\phi(x)' = g\phi(x) = g\xi(\varphi(x))\tilde{\phi}(x). \tag{2.6}$$

Since $g\xi(\varphi(x))$ is also an element of G, it must be in some right coset with the representative $\xi(\varphi'(x))$

$$g\xi(\varphi(x)) = \xi(\varphi'(x))h(\varphi(x), g), \tag{2.7}$$

where $h(\varphi(x), g)$ is some element of H that depends both on φ and g, with $h(\varphi(x), h) = h$. From (2.7) the transformation properties of $\tilde{\phi}(x)$ follow trivially

$$\tilde{\phi}(x)' = h(\varphi(x), g)\tilde{\phi}(x). \tag{2.8}$$

The transformation properties of $\varphi(x)$ and $\tilde{\phi}(x)$ under G are in general very complicated and far from linear. It is thus of great use if one is able to construct functions of Goldstone boson fields and/or other fields that do transform linearly under G. These fields will be constructed explicitly for the case of $SU(3)_R \times SU(3)_L$ in the next section.

Let us now discuss the construction of the effective Lagrangian $\mathcal{L}_{\text{eff}}(\tilde{\phi}, \varphi)$ from fields $\tilde{\phi}(x)$ and $\varphi(x)$. The effective Lagrangian is assumed to be invariant under G. Since in general neither $\tilde{\phi}(x)$ nor $\xi(\varphi(x))$ transform linearly under G, one could expect that the symmetry properties of the effective Lagrangian $\mathcal{L}_{\text{eff}}(\tilde{\phi}, \varphi)$ could be somewhat distorted in the transition to the description with the corresponding effective action $\Gamma[\tilde{\phi}, \varphi]$. This is not the case as can be shown by the following argument. We start with the element of the Lie algebra of G

$$\xi^{-1}(\varphi(x))\partial_\mu \xi(\varphi(x)) = i \sum_a x_a D_{a\mu}(x) + i \sum_i t_i E_{i\mu}(x), \tag{2.9}$$

with

$$D_{a\mu}(x) = \sum_b D_{ab}(\varphi(x))\partial_\mu \varphi_b(x), \tag{2.10a}$$

$$E_{i\mu}(x) = \sum_b E_{ib}(\varphi(x))\partial_\mu \varphi_b(x), \tag{2.10b}$$

where t_i denote the generators of Lie algebra of H, while x_a denote the other generators in Lie algebra of G, as before. It is then fairly easy to show (see chapter 19 of [?]) that $D_{a\mu}(x)$ and $\mathcal{D}_\mu \tilde{\phi}(x) = \partial_\mu \tilde{\phi}(x) + i \sum_j t_j E_{j\mu}(x)\tilde{\phi}(x)$ transform as

$$D_{a\mu}(x)' = \sum_b T_{ab}(h(\varphi(x), g)) D_{b\mu}(x), \tag{2.11}$$

$$\left(\mathcal{D}_\mu \tilde{\phi}(x)\right)' = h(\varphi(x), g)\mathcal{D}_\mu \tilde{\phi}(x), \tag{2.12}$$

with $T_{ab}(h)$ the adjoint representation of H. Note that $h(\varphi(x), h) = h$, so that $D_{a\mu}(x)$, $\mathcal{D}_\mu \tilde{\phi}(x)$ in (2.11), (2.12) transform linearly under H. Note also, that $\xi(\varphi(x))$ cannot appear explicitly in the effective Lagrangian $\mathcal{L}_{\text{eff}}$, because this is assumed to be invariant under the global G transformations and the $\xi(\varphi(x))$ factors can be rotated away. Consider for instance the derivative

$$\partial_\mu \left(\xi(\varphi(x))\tilde{\phi}(x)\right) = \xi(\varphi(x)) \left(\partial_\mu \tilde{\phi}(x) + \xi^{-1}(x)\partial \xi(x)\tilde{\phi}(x)\right). \tag{2.13}$$

The factor $\xi(\varphi(x))$ on the right-hand side can then be rotated away. In the effective Lagrangian $\mathcal{L}_{\text{eff}}$ only terms with at least one derivative on the Goldstone boson fields or with no Goldstone boson fields will remain. The most general effective Lagrangian $\mathcal{L}_{\text{eff}}$ invariant under G can then be constructed from $\tilde{\phi}$, $D_{a\mu}$, $\mathcal{D}_\mu \tilde{\phi}$, $\mathcal{D}_\mu \mathcal{D}_\nu \tilde{\phi}$,.... by constructing the most general expression invariant under H. Such Lagrangian will then *be invariant under the full group* G also. Since these combinations of fields transform linearly under H, the effective action Γ will be invariant under H as well. Chiral counting[2] prohibits the terms with no derivative on Goldstone fields to appear in the effective action Γ. The effective action Γ will then be an expression constructed from $\tilde{\phi}$, $D_{a\mu}$, $\mathcal{D}_\mu \tilde{\phi}$, $\mathcal{D}_\mu \mathcal{D}_\nu \tilde{\phi}$,...., invariant under H and thus under G.

Incidentally, the argument presented above insures also the renormalizability (as understood in the general sense of the word) of the effective Lagrangian approach. No terms that are not already present in the effective Lagrangian $\mathcal{L}_{\text{eff}}$ can appear in the effective action Γ. All divergences that appear in the course of the calculation can be reabsorbed into the definitions of the couplings appearing in the effective Lagrangian $\mathcal{L}_{\text{eff}}$.

[2]This is explained at the end of section 2.3.

2.2 Chiral perturbation theory

One of the earliest and also one of the most successful examples of effective theories is the chiral perturbation theory (χPT) which we will briefly review in this section. The expansion parameter in the chiral perturbation theory is the momentum exchange in the process, p^2. Argument for the validity of this expansion will be given at the end of the next section.

To start with, let us write down the QCD Lagrangian (we neglect the weak interactions in the following)

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_n \bar{\Psi}^{(n)} \big(i\gamma^\mu D_\mu - m^{(n)} \big) \Psi^{(n)}, \tag{2.14}$$

where the summation is over different quark flavors, $\Psi^{(n)}$ are quark fields and the covariant derivative is

$$D_\mu \Psi^{(n)} = \big(\partial_\mu + i g_s G^a_\mu T_a \big) \Psi^{(n)}, \tag{2.15}$$

with G^a_μ the gluon field, g_s the strong coupling and $T_a = \lambda^a/2$ with λ_a the Gell-Mann $SU(3)$ matrices, for which $\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab}$. The gauge field strength tensor (curvature tensor) is

$$T_a F^a_{\mu\nu} \Psi = [D_\mu, D_\nu] \Psi. \tag{2.16}$$

Let us now focus only on the three light quark flavors u, d, s. As these quarks are relatively light (with quark masses small compared to the chiral scale as we will see below) the Lagrangian (2.14) is approximately invariant under the left and right chiral rotations $U(3)_R \times U(3)_L$

$$\Psi'^n = \Big[\exp\big(i T_a \theta^a_R \tfrac{1}{2}(1+\gamma_5) \big) \Big]^{nm} \Big[\exp\big(i T_a \theta^a_L \tfrac{1}{2}(1-\gamma_5) \big) \Big]^{ml} \Psi^l. \tag{2.17}$$

The axial $U(1)_A$ is anomalous and is broken by nonperturbative effects. The vector $U(1)_V$ is the global symmetry group of the baryon number and is not needed for further discussion. In the following we then focus on the $SU(3)_R \times SU(3)_L$ global transformation group that is assumed to be spontaneously broken down to the vector subgroup $SU(3)_V$. Following the general procedure outlined in the previous section we define the quark fields with the Goldstone bosons factored out

$$\Psi = e^{i\gamma_5 \varphi(x)} \tilde{\Psi}. \tag{2.18}$$

The Goldstone boson fields $\varphi(x)$ then transform under the $SU(3)_R \times SU(3)_L$ according to (cf. (2.7))

$$e^{iT_a\theta^a_R \frac{1}{2}(1+\gamma_5)} e^{iT_a\theta^a_L \frac{1}{2}(1-\gamma_5)} e^{i\gamma_5\varphi(x)} = e^{i\gamma_5\varphi'(x)} e^{iT_a\theta^a_V(\varphi(x))}, \tag{2.19}$$

where $\varphi'(x)$ is a 3×3 matrix of transformed Goldstone boson fields, while the dependence of $\theta^a_V(\varphi(x))$ on the parameters of global $SU(3)\times SU(3)$ transformations, $\theta^a_{L,R}$, has not been denoted explicitly in (2.19). Multiplying with projectors $\frac{1}{2}(1\pm\gamma_5)$ one arrives at

$$e^{iT_a\theta^a_R} e^{i\varphi(x)} = e^{i\varphi'(x)} e^{iT_a\theta^a_V(\varphi(x))}, \tag{2.20}$$
$$e^{iT_a\theta^a_L} e^{-i\varphi(x)} = e^{-i\varphi'(x)} e^{iT_a\theta^a_V(\varphi(x))}. \tag{2.21}$$

Following [?] we introduce $R = \exp\left(iT_a\theta_R^a\right)$, $L = \exp\left(iT_a\theta_L^a\right)$, $U(x) = \exp\left(iT_a\theta_V^a(\varphi(x))\right)$, $\Sigma = \exp(2i\varphi(x))$. The transformation properties of Σ are then

$$\Sigma' = R\Sigma L^\dagger. \tag{2.22}$$

The Σ field thus transforms linearly under the global transformations. The quark fields with Goldstone fields factored out transform as

$$\tilde{\Psi}' = \exp\left(iT_a\theta_V^a(\varphi(x))\right)\tilde{\Psi} = U(x)\tilde{\Psi}. \tag{2.23}$$

To make contact with the previous section we introduce also $\xi(x) = \exp\left(i\varphi(x)\right)$ and vector and axial vector fields $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ given by:

$$\mathcal{V}_\mu = \frac{1}{2}(\xi\partial_\mu\xi^\dagger + \xi^\dagger\partial_\mu\xi) \qquad\qquad \mathcal{A}_\mu = \frac{i}{2}(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger), \tag{2.24}$$

They transform according to (2.20), (2.21) as

$$\mathcal{V}'_\mu = U\mathcal{V}_\mu U^\dagger + U\partial_\mu U^\dagger \qquad\qquad \mathcal{A}'_\mu = U\mathcal{A}_\mu U^\dagger. \tag{2.25}$$

The axial vector $\mathcal{A}_\mu$ and vector $\mathcal{V}_\mu$ currents are (apart from the constant factor) exactly the $\sum_a x_a D_{a\mu}(x)$ and $\sum_i t_i E_{i\mu}(x)$ parts of (2.9) respectively. In other words, the axial vector current $\mathcal{A}_\mu$ is the $D_{a\mu}$ of (2.11), while the covariant derivative of (2.12) is $(\partial_\mu + \mathcal{V}_\mu)\tilde{\Psi}$.

Up to this point we have been assuming that the $SU(3)_R \times SU(3)_L$ is an exact symmetry of the QCD Lagrangian. However, this symmetry is broken by the mass term in (2.14). To introduce the breaking in the effective Lagrangian it is useful to introduce an external field χ that in the end is set equal to the value of the mass matrix $m = m^{(n)}\delta_{nn'}$. We make the replacement [?]

$$\bar{\Psi}m\Psi = \bar{\Psi}_R m\Psi_L + \bar{\Psi}_L m\Psi_R \to \bar{\Psi}_R\chi\Psi_L + \bar{\Psi}_L\chi^\dagger\Psi_R. \tag{2.26}$$

If the external field χ is assumed to transform according to $\chi \to R\chi L^\dagger$, the "corrected" QCD Lagrangian is then invariant under the $SU(3)_R \times SU(3)_L$. This will then be true also for the effective Lagrangian.

Before we write down the final expression for the leading order Lagrangian in the chiral expansion, we rescale the Goldstone boson fields $\varphi = \Pi/f$, where Π is a 3×3 traceless matrix of light pseudoscalar fields

$$\Pi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}, \tag{2.27}$$

and f is a dimensionful parameter that is determined from experiment. At the leading order it is equal to the pion decay constant $f_\pi = 132$ MeV. Later on, in section 2.5, we will also use the notation $\Pi = \sum_i P^i t^i$, with $t^i = \lambda^i/\sqrt{2}$, where λ^i are the $SU(3)$ Gell-Mann matrices.

Since Π/f are Goldstone boson fields, the effective Lagrangian does not contain terms without derivatives on Π, as explained at the end of the previous section (see the discussion below Eq. (2.13)). For the low energy processes the momentum exchange p^2 can then be used as an expansion parameter. In the leading order chiral Lagrangian only the terms with the smallest number of derivatives are kept. Using the counting $p^2 \sim m_q \sim m_\pi^2$, one arrives at the usual $\mathcal{O}(p^2)$ chiral Lagrangian for the light pseudoscalar mesons [?]

$$\mathcal{L}_{\rm str}^{(2)} = \frac{f^2}{8}\,\mathrm{tr}(\partial^\mu\Sigma\partial_\mu\Sigma^\dagger) + \frac{f^2\mu_0}{2}\,\mathrm{tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi), \tag{2.28}$$

where $\Sigma = \exp(2i\Pi/f)$ with Π given in (2.27), while the trace tr runs over flavor indices. The external field χ is set in the calculation equal to the current quark mass matrix $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$. The coefficient of the first term in (2.28) is fixed by the requirement that the kinetic term of pseudoscalar mesons is properly normalized. The second term on the other hand contains an additional unknown constant μ_0. This terms leads at the leading order to the Gell-Mann–Oakes–Renner relations [?] $m_\pi^2 \sim 2\mu_0(m_u + m_d)$, $m_K^2 \sim 2\mu_0(m_{u,d} + m_s)$, $m_\eta^2 \sim \frac{2}{3}\mu_0(m_u + m_d + 4m_s)$.

The order $\mathcal{O}(p^4)$ Lagrangian contains ten additional terms [?], which we refer to as counterterms. We will write down explicitly only the terms that contribute to the π and K wave function renormalization factors and to the f_π, f_K decays constants (cf. section 2.5). Other counterterms will not enter our analysis. The relevant terms are

$$\mathcal{L}_4 = L_4 4\mu_0\,\mathrm{tr}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)\,\mathrm{tr}(\mathcal{M}\Sigma^\dagger + \Sigma\mathcal{M}^\dagger) + L_5 4\mu_0\,\mathrm{tr}(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma[\mathcal{M}\Sigma^\dagger + \Sigma\mathcal{M}^\dagger]) + \ldots \tag{2.29}$$

while the complete $\mathcal{O}(p^4)$ Lagrangian can be found in [?].

From the chiral Lagrangian one can also deduce the form of the light weak current (see e.g. [?]). At the order $\mathcal{O}(p)$ this is

$$j_\mu^a = -i\frac{f^2}{4}\,\mathrm{tr}(\Sigma\partial_\mu\Sigma^\dagger\lambda^a), \tag{2.30}$$

corresponding to the quark current $j_\mu^a = \bar{q}_L\gamma_\mu\lambda^a q_L$, with λ^a an SU(3) flavor matrix.

2.3 Heavy Quark Effective Theory and Chiral Expansion

Since its early applications [?] the heavy quark symmetry has been one of the key ingredients in the theoretical investigations of hadrons containing a heavy quark. It has been successfully applied to the heavy hadron spectroscopy, to the inclusive as well as to a number of exclusive decays (for reviews of the heavy quark effective theory and related issues see [?] or [?]). To describe interactions with not too energetic light mesons, the heavy quark symmetry has been combined with chiral symmetries leading to the heavy hadron chiral perturbation theory (HHχPT) [?].

The important observation in the heavy quark expansion is that the mesons containing an infinitely heavy quark Q exhibit a set of simple properties. Since a heavy quark is very massive its Compton wavelength is much smaller than the size of the meson. The latter is determined by the wave function corresponding to the light degrees of freedom, the light quarks and the soft gluons. In the limit of an infinitely heavy quark, the wave function of the light degrees of freedom is the solution of QCD field equations for a static triplet color source. It is thus independent of the spin of the heavy quark as well as of its flavor. That is, the solution for the light degrees of freedom does not change if we replace $Q(v,s)$ with $Q'(v,s')$, where v and s denote velocity and spin of the heavy quark respectively.

To get more quantitative, let us consider a hadron with a heavy quark. The major part of the momentum is carried by the heavy quark. This propagates almost unperturbed and interacts with light degrees of freedom only through small exchanges of momenta. In words of Neubert: “The heavy quark flies like a rock!" [?]. It is thus useful to separate the heavy quark momentum P_Q into the momentum due to the movement of the meson $m_Q v$ and the perturbations

$$P_Q^\mu = m_Q v^\mu + k^\mu, \tag{2.31}$$

where v^μ is the four-velocity of the hadron. The heavy quark propagator is then

$$\frac{i}{\not{P}_Q - m_Q + i\epsilon} = \frac{i}{v\cdot k + i\epsilon}\frac{1+\not{v}}{2} + \mathcal{O}(k/m_Q) \to \frac{i}{v\cdot k + i\epsilon}P_+, \tag{2.32}$$

where in the last step the limit $m_q \to \infty$ has been taken and the projectors $P_\pm = (1\pm\not{v})/2$ have been introduced. Since $P_+\gamma^\mu P_+ = v^\mu$ also the couplings of the heavy quark to gluons $g_s T_a \gamma^\mu$ (2.14) can be simplified to $g_s T_a v^\mu$ at the leading order in $1/m_Q$. The Lagrangian corresponding to these Feynman rules is

$$\mathcal{L} = \bar{h}_v(iv^\mu\partial_\mu - g_s v^\mu G_\mu^a T_a)h_v, \tag{2.33}$$

where h_v satisfies $P_+h_v = h_v$, $P_-h_v = 0$. This Lagrangian can be obtained from the QCD Lagrangian (2.14) by projecting to the "large Dirac components" and factoring out the trivial phase change due to the hadron movement

$$h_v \sim P_+ e^{im_Q v\cdot x} Q. \tag{2.34}$$

Neglecting terms suppressed by additional powers of $1/m_Q$ this replacement leads to the Lagrangian (2.33). The heavy quark Lagrangian exhibits the heavy quark spin symmetry. Intuitively this can be expected from the fact that no Dirac gamma matrices appear in (2.33). Formally, it is easy to show that the Lagrangian (2.33) is invariant under the generators of $SU(2)$ transformations $S^i = \frac{1}{2}\gamma_5\not{v}\not{e}^i$, where $i = 1,2,3$, while e^i are three vectors orthogonal to the heavy quark velocity v, $v\cdot e = 0$. For S^i then[3]

$$[S^i, S^j] = i\epsilon^{ijk}S^k, \qquad [\not{v}, S^i] = 0, \tag{2.35}$$

and $S_i^\dagger = \gamma_0 S_i \gamma_0$, from which it trivially follows that the Lagrangian (2.33) does not change under the transformation

$$h_v' = (1 + i\theta^i S^i)h_v, \tag{2.36}$$

with both h_v' and h_v satisfying $P_-h_v' = P_-h_v = 0$.

To be able to construct the effective Lagrangian on the meson level, we have to consider the transformation properties of the heavy mesons under Lorentz, heavy quark spin and flavor symmetries. We will follow the elegant tensor representation formalism [?, ?], but constrain ourselves only to the case of $J^P = 0^-, 1^-$ mesons.

The mesons consist of the heavy quark Q and a light antiquark $\bar{q}_a$. These are described by the Dirac spinor field h_v for which $\not{v}h_v = h_v$ and the light quark field $\bar{q}_a$ for which $\bar{q}_a\not{v} = -\bar{q}_a$. The minus sign in the last equation is necessary in order to project out predominantly the antiquark degrees of freedom. The meson field will then be represented by a Dirac spinor-antispinor field $H_a \sim h_v\bar{q}_a$ that in general has 16 components. However the requirements

$$\frac{1-\not{v}}{2}H_a = 0, \qquad H_a\frac{1+\not{v}}{2} = 0, \tag{2.37}$$

[3] To prove these relations it is best to go to the heavy hadron rest frame [?].

reduce the 16 components to only 4 independent components (each of the projectors reduces a Dirac bispinor to a two-component spinor). The four components will then describe the heavy pseudoscalar meson with one entry and the vector meson with three independent degrees of freedom.

The most general field satisfying requirements (2.37) is then

$$H_a = \frac{1+\not{v}}{2}\left(P_a^\mu \gamma_\mu - P_a \gamma_5\right), \tag{2.38}$$

with

$$v^\mu P_\mu = 0. \tag{2.39}$$

As expected the P_a is the pseudoscalar field and P_a^μ the vector meson field. The transformation of the heavy meson field (2.38) under the heavy quark spin transformations (2.36) is then

$$H_a \to (1 + i\theta^i S^i) H_a. \tag{2.40}$$

To take into considerations also the interactions with the Goldstone bosons these are factored out from H_a as outlined in sections 2.1, 2.2 (i.e. the $\bar{q}_a$ is replaced by $\bar{\tilde{q}}_a$, that transforms according to (2.23)). Under the chiral $SU(3)_R \times SU(3)_L$ transformations thus

$$H_a \to H_b U_{ba}^\dagger, \tag{2.41}$$

where we did not write the tilde on the H_a field. Finally, under Lorentz transformations Λ, the H_a field transforms as $h_v \bar{q}_a$

$$H_a \to D(\Lambda) H_a D^{-1}(\Lambda), \tag{2.42}$$

with $D(\Lambda)$ the $(1/2,0) \oplus (0,1/2)$ representation of the Lorentz group $D(\Lambda) = \exp[i\frac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu}]$.

The most general effective Lagrangian to order $\mathcal{O}(p)$ in the chiral expansion, that is invariant under the transformations (2.40), (2.41), taking into account the restrictions (2.37), and is a Lorentz scalar, is [?]

$$\mathcal{L}_{\text{str}}^{(1)} = -\operatorname{Tr}(\bar{H}_a iv\cdot D_{ab} H_b) + g \operatorname{Tr}(\bar{H}_a H_b \gamma_\mu \mathcal{A}_{ba}^\mu \gamma_5), \tag{2.43}$$

where $D_{ab}^\mu H_b = \partial^\mu H_a - H_b \mathcal{V}_{ba}^\mu$, while the trace Tr runs over Dirac indices. Note that in (2.43) and the rest of this section a and b are *flavor* indices.

The vector and axial vector fields $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ in (2.43) are the same as in (2.24) and are given by:

$$\mathcal{V}_\mu = \frac{1}{2}(\xi\partial_\mu\xi^\dagger + \xi^\dagger\partial_\mu\xi) \qquad\qquad \mathcal{A}_\mu = \frac{i}{2}(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger), \tag{2.44}$$

where $\xi = \exp(i\Pi/f)$, with Π defined in (2.27). The $\overline{H}_a$ field is $\overline{H}_a = \gamma_0 H_a^\dagger \gamma_0$, with H_a defined in (2.38).

The higher order terms (referred to as counterterms) in the expansion in $v \cdot p \sim \mathcal{O}(p)$ and $m_q \sim \mathcal{O}(p^2)$ are then up to the order $\mathcal{O}(p^3)$

$$\begin{aligned}\mathcal{L}_3^{\text{heavy}} =& 2\lambda_1 \operatorname{Tr}[\bar{H}_a H_b](\mathcal{M}_+)_{ba} + k_1 \operatorname{Tr}[\bar{H}_a iv\cdot D_{bc} H_b](\mathcal{M}_+)_{ca} \\ &+ k_2 \operatorname{Tr}[\bar{H}_a iv\cdot D_{ba} H_b] \operatorname{tr}(\mathcal{M}_+) + \delta\mathcal{L}_g \ldots\end{aligned} \tag{2.45}$$

with $\mathcal{M}_+ = \frac{1}{2}(\xi^\dagger M \xi^\dagger + \xi M \xi)$ and $\delta\mathcal{L}_q$ given in (**??**). Dots denote terms that were not written out, as they do not contribute to the Z_D, Z_{D_s} wave function renormalization factors and the f_D, f_{D_s} decay constants that will be discussed in section 2.5 nor to the decays $D^0 \to K^0 \bar{K}^0$, $D^0 \to \gamma\gamma$, $D^0 \to l^+ l^- \gamma$ considered in chapters 5 and **??**. The effect of λ_1 term is to change the heavy meson propagator. In the case of s quark the shift is $v \cdot p \to v \cdot p - \Delta$, where $\Delta = m_{H_s} - m_H$. The k_1, k_2 terms will contribute to the wave function renormalization of the heavy mesons. Note that we did not include in the analysis the terms suppressed by $1/m_H$. These are considered to be of higher order in the expansion.

The $\delta\mathcal{L}_g$ part of the Lagrangian (2.45) will contribute a correction to the $D^* D\pi$ coupling from which the value of g will be obtained. Neglecting $1/m_H$ terms, one gets [**?**, **?**]

Next we consider bosonization of currents that appear in weak decays. At the leading order in $1/m_H$ and at the next-to-leading order in chiral expansion this is [**?**, **?**]

$$\begin{aligned}\bar{q}_a \gamma^\mu (1-\gamma_5) Q \to & \frac{i\alpha}{2} \operatorname{Tr}[\gamma^\mu (1-\gamma_5) H_b] \xi^\dagger_{ba} \\ & + \frac{i\alpha}{2} \varkappa_1 \operatorname{Tr}[\gamma^\mu (1-\gamma_5) H_c] \xi^\dagger_{ba} (\mathcal{M}_+)_{cb} \\ & + \frac{i\alpha}{2} \varkappa_2 \operatorname{Tr}[\gamma^\mu (1-\gamma_5) H_b] \xi^\dagger_{ba} \operatorname{tr}(\mathcal{M}_+) + \dots\end{aligned} \tag{2.46}$$

Beside the leading order $\mathcal{O}(p^0)$ current in the chiral counting, given in the first line of (2.46), we display also two $\mathcal{O}(p^2)$ terms. These will be relevant for the discussion of the f_D, f_{D_s} decay constants given later on in section 2.5. Other $\mathcal{O}(p^2)$ terms are not written out explicitly. They can be found in [**?**].

In the same way as the heavy-light current (2.46), operators of more general structure $(\bar{q}_a \Gamma Q)$, with Γ an arbitrary product of Dirac matrices, can be translated into an operator containing meson fields only [**?**]. At the leading order

$$(\bar{q}_a \Gamma Q) \to \frac{i\alpha}{2} \operatorname{Tr}[P_R \Gamma H_b \xi^\dagger_{ba}] + \frac{i\alpha}{2} \operatorname{Tr}[P_L \Gamma H_b \xi_{ba}]. \tag{2.47}$$

For instance the operator $(\bar{q}_a \sigma_{\mu\nu} \frac{1}{2}(1+\gamma_5) Q)$ proportional to Q_7 operator (cf. section (4.2)) is then translated into

$$\left(\bar{q}_a \sigma_{\mu\nu} \tfrac{1}{2}(1+\gamma_5) Q\right) \to \frac{i\alpha}{2} \operatorname{Tr}\left[\sigma_{\mu\nu} \tfrac{1}{2}(1+\gamma_5) H_b \xi^\dagger_{ba}\right]. \tag{2.48}$$

Finally, let us give the Weinberg's counting rule [**?**] for the case of heavy hadron chiral perturbation theory. This counting rule establishes the relative importance of loop contributions. Consider a general diagram with L loops, for which we do the counting in terms of momenta $p \ll 4\pi f$ that flow in the internal lines. The Goldstone boson propagators are of the form $1/(p^2 - m_\pi^2)$ and contribute a factor of p^{-2} for each propagator. Similarly the heavy meson propagators $\sim 1/v \cdot p$ contribute a factor of p^{-1}. Each loop contributes an integration factor $d^4 p$, so that finally the dimension of the diagram in terms of p is

2.4 Photon couplings and gauge invariance

The photon couplings are obtained by gauging the Lagrangians (2.28), (2.43) and the light current (2.30) with the $U(1)$ photon field B_μ. The covariant derivatives are then $\mathcal{D}^\mu_{ab} H_b = \partial^\mu H_a + ieB^\mu (Q'H - HQ)_a - H_b \mathcal{V}^\mu_{ba}$ and $\mathcal{D}_\mu \xi = \partial_\mu \xi + ieB_\mu [Q, \xi]$ with $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and Q' the heavy quark charge ($Q' = \frac{2}{3}, -\frac{1}{3}$ for the case of c and b quarks respectively). The vector and axial vector fields (2.24) change after gauging and are $\mathcal{V}_\mu = \frac{1}{2}(\xi \mathcal{D}_\mu \xi^\dagger + \xi^\dagger \mathcal{D}_\mu \xi)$ and $\mathcal{A}_\mu = \frac{i}{2}(\xi^\dagger \mathcal{D}_\mu \xi - \xi \mathcal{D}_\mu \xi^\dagger)$. Similarly, the light weak current (2.30) contains after gauging the covariant derivative $\mathcal{D}_\mu$ instead of ∂_μ. However, the gauging procedure alone does not introduce a coupling of the form $D^* D \gamma$ without emission of additional Goldstone bosons. To describe this electromagnetic interaction we follow [?] introducing an additional gauge invariant contact term with an unknown coupling β of dimension -1.

$$\mathcal{L}_\beta = -\frac{\beta e}{4} \operatorname{Tr} \bar{H}_a H_b \sigma^{\mu\nu} F_{\mu\nu} Q^\xi_{ba} - \frac{e}{4 m_Q} Q' \operatorname{Tr} \bar{H}_a \sigma^{\mu\nu} H_a F_{\mu\nu}, \tag{2.49}$$

where $Q^\xi = \frac{1}{2}(\xi^\dagger Q \xi + \xi Q \xi^\dagger)$ and $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The first term concerns the contribution of the light quarks in the heavy meson and the second term describes emission of a photon from the heavy quark. Its coefficient is fixed by the heavy quark symmetry. From (2.49) both $H^* H \gamma$ and $H^* H^* \gamma$ interaction terms arise. Even though the Lagrangian (2.49) is formally of higher order in $1/m_Q$ or chiral expansion, we do not neglect it, as it has been found that it gives a sizable contribution to $D^*(B^*) \to D(B)\gamma\gamma$ decays [?]. In chapter **??** we will find, that in the $D^0 \to \gamma\gamma$ decay the Lagrangian terms (2.49) give the largest contribution to the parity conserving part of the amplitude. However, they do not contribute to the decay rate by more than 10%. The Lagrangian (2.49) in principle receives a number of other contributions at the order $\mathcal{O}(1/m_Q)$, but these can be absorbed in the definition of β for the processes $D^0 \to \gamma\gamma$, $D^0 \to l^+ l^- \gamma$, that will be considered in chapter **??** [?].

In the following we present two proofs that such gauging procedure of the effective Lagrangian does indeed lead to a gauge invariant effective action and thus to a gauge invariant amplitude. The general proof is just a special case of the proof given in chapter 16 of [?], that has already been cited in section 2.1 (cf. Eq. (2.2)). The electromagnetic $U(1)$ transformations of fields appearing in the effective Lagrangians (2.28), (2.43) are linear

$$H_a \to e^{ieQ'\alpha(x)} H_a e^{-ieQ_a \alpha(x)} = H_a + ie(Q'H - HQ)_a \alpha(x) + \cdots \tag{2.50}$$

$$\xi \to e^{ieQ\alpha(x)} \xi e^{-ieQ\alpha(x)} = \xi + ie[Q, \xi]\alpha(x) + \cdots \tag{2.51}$$

$$B_\mu \to B_\mu - \partial_\mu \alpha, \tag{2.52}$$

with Q_a the Q_{aa} component of $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and no summation over a. The cited proof then states that as long as the effective Lagrangian is gauge invariant under the linear transformations (2.50)-(2.52), so is the effective action, which is what we wanted to show.

The general proof does not help us in the calculation, where one wishes to find *finite* sets of diagrams, that are already gauge invariant. Here a very useful tool is a diagrammatic proof of gauge invariance, which we state next. Consider an arbitrary off-shell initial Feynman diagram with arbitrary number of loops, heavy lines and photon lines. The sum of the diagrams obtained by inserting an additional photon line everywhere in the initial diagram, where this is permitted by the gauged Lagrangians (2.28), (2.43) and (2.49), is gauge invariant. Finding gauge invariant sets of diagrams in the actual calculation is then straightforward. One starts from an appropriate initial diagram, inserts photon vertices everywhere and ends up with a gauge invariant set.

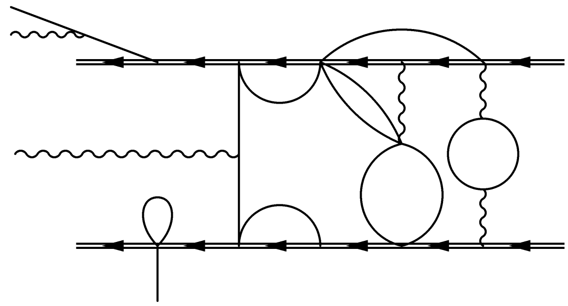


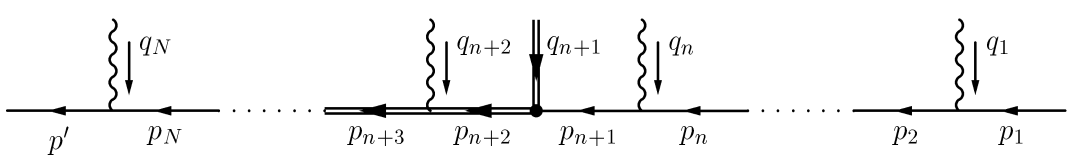
Figure 2.3: Example of an initial Feynman diagram. The gauge invariant set is obtained by adding a photon leg wherever this is possible. The heavy mesons are depicted by the double lines, while light pseudoscalars are represented by the solid lines.

The proof of the above statement follows closely the proof of gauge invariance of QED amplitudes as presented in the textbook of Peskin and Schroeder [?]. The complication is, that we have to deal with two sorts of charged particles, the heavy mesons and light-pseudoscalars, and with an in principle infinite number of couplings between them. We shall prove the statement about gauge invariance only for the vertices with up to three pseudoscalar and/or heavy meson fields, as this will be needed further on in the calculations done in the thesis. At the end we shall present also a discussion concerning more general vertices.

The expressions for the vertices follow from the effective Lagrangians (2.28), (2.43). For the coupling of photon(s) to the light pseudoscalars, the coupling is of the form

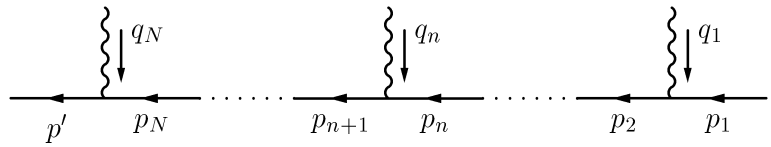
Let us now consider an arbitrary Feynman diagram with incoming and outgoing legs off-shell, where we limit ourselves to the case of couplings (**??**)-(**??**). Since there are only up to three mesons per each vertex, only two of the meson fields can be charged. To each vertex we can thus associate a charged line with one ingoing and one outgoing charged meson leg and thus with a well defined direction of charge flow. The initial diagram is interlaced with such charged lines. Since to each vertex only one charged line is associated, the charged lines never cross. In other words, to a given charged line only neutral lines attach. Because charged lines are connected only by neutral lines, each charged line can be considered separately.

Charged line can either form a loop or connect to two external charged legs. To begin with we consider the charged line that begins and ends on the external off-shell charged legs. Such a line of charge flow has a general structure



with double lines representing the heavy mesons, and the solid lines denoting the light pseudoscalars. Lines are arranged so that the charged lines are horizontal with neutral lines attached to it (i.e. the heavy meson carrying momentum q_{n+1} is neutral).

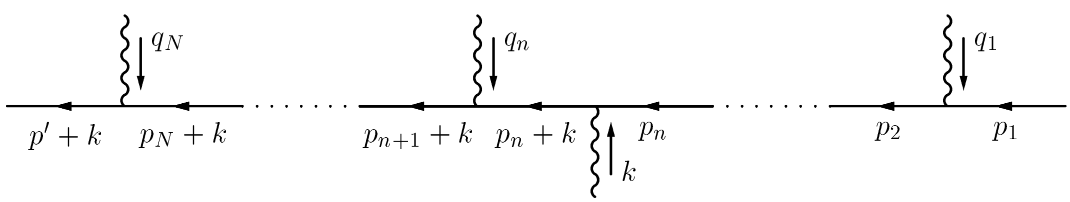
To simplify the problem even further, we shall first consider the charged line of only light pseudoscalar mesons, with coupling to photons given by (**??**). For simplicity we also assume, that to this initial charged light pseudoscalar line only single photon lines attach



The part of the amplitude corresponding to this initial charged line is then of the form

In the next step we attach an additional external photon line to the initial charged line, wherever this is possible. The Lorentz index μ of the additional external photon line carrying incoming momentum k is contracted by the polarization vector $\epsilon^\mu(k)$, when the photon line is put on-shell. A gauge invariant amplitude has to be invariant under the change $\epsilon^\mu(k) + ck^\mu$. To test gauge invariance we thus contract the Lorentz index μ of the additional external photon line with k^μ. This should then give vanishing result for the corresponding amplitude, when external legs are put on-shell.

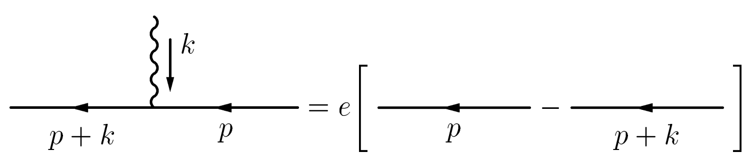
The additional photon line can be either attached to the pseudoscalar propagator or to the vertex already containing one photon leg (**??**). When we attach the photon to the $n-$th propagator, all the momenta in the propagators after it get shifted by k



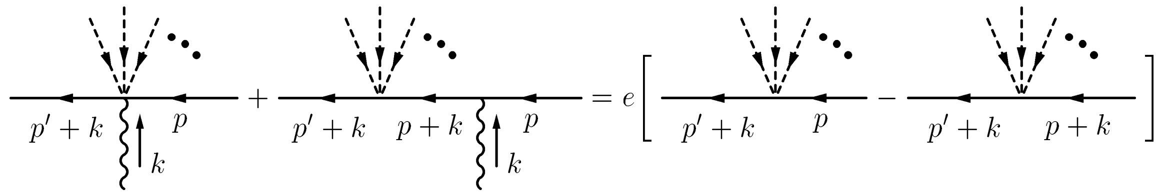
The invariant amplitude corresponding to the above diagram is then

Similar reasoning applies if the line is closed, i.e., if the charged line forms a loop. Then the final amplitude involves also the integration over loop variable, so that

In the reasoning outlined above there were two crucial steps. First, the propagator identity



has to hold for any charged particle. And second, for each amputated vertex multiplied only by the propagator next to it to the right, the following identity has to be true



where an arbitrary number of neutral lines (shown as dashed lines) attach to the vertex. The first identity is needed so that also the photon attached to the last leg in the charged line can be represented as a difference of two propagators.

Let us first prove that the propagator identity holds for the mesons involved in the problem. For the light pseudoscalars we have

The diagram 2) is

We would like to show, that the sum of (**??**) and (**??**) is

For the vertices with more than two charged lines and more than one derivative on some of the charged fields, additional complications arise. First of all the charge flow lines can now cross each other. Because of the crossing, charged lines are not uniquely defined. In fact, to prove gauge invariance, one has to consider all possible ways of defining charge flow lines. To deal with this, one focuses on one charged line only, defining also which fields in the Lagrangian destroy/create legs of this line and regard other charged lines attached to the chosen charge flow as we did the neutral legs before. If there are not more than one derivative acting on each meson field, everything proceeds as it did above.

Extension of the arguments given above to the case of more than one derivative acting on the charged fields in the Lagrangian is not straightforward. In this case also the propagator identities (counterparts of the Eqs. (**??**)-(**??**)) change and become more complex. The theorem is then easiest to prove on a case by case basis.

2.5 Determination of the parameters

The unknown couplings appearing in the Lagrangians (2.28), (2.43) are obtained from experiment. In the following we shall present the determination of the couplings first at tree level and then also at the one loop level.

Tree level

At tree level α (2.46) is trivially related to the heavy meson decay constant $f_{B,D} = \alpha/\sqrt{m_{B,D}}$, where the decay constant is defined through an axial current matrix element. For e.g f_D this is

$$\langle 0|\overline{u}\gamma^\mu\gamma_5 c|D^0\rangle = ip_D^\mu f_D, \tag{2.53}$$

From [**?**, **?**] one deduces $f_{D_s} = 268 \pm 25$ GeV, from which $\alpha^{\mathrm{Tree}} = 0.38 \pm 0.04\ \mathrm{GeV}^{3/2}$. Note that this value has been extracted from a system with a valence s-quark and one expects a sizable 1-loop correction.

From the CLEO measurement of the $D^{*+} \to D^0\pi^+$ partial decay width [?, ?], the value of $g^{\text{Tree}} = 0.59 \pm 0.08$ can be deduced, with g^{Tree} being the $D^*D\pi$ coupling constant (cf. Eq. (2.43)). The pion decay constant is taken to be $f^{\text{Tree}} = f_\pi = 130.7 \pm 0.4$ MeV [?].

In order to obtain the value of β we use the available experimental data from $D^{*+} \to D^+\gamma$ and $D^{*0} \to D^0\gamma$ decays. For instance, one can use the recently determined D^{*+} decay width $\Gamma(D^{*+}) = 96 \pm 4 \pm 22$ keV [?, ?] together with the branching ratio $\text{Br}(D^{*+} \to D^+\gamma) = (1.6 \pm 0.4)\%$ [?]. At tree level one has

$$\Gamma(D^{*+} \to D^+\gamma) = \frac{e^2}{12\pi}\left(\frac{2}{3}\frac{1}{m_c} - \frac{1}{3}\beta\right)^2 k_\gamma^3, \tag{2.54}$$

with $k_\gamma = \frac{m_{D^*}}{2}(1 - \frac{m_D^2}{m_{D^*}^2})$ the momentum of the outgoing photon. Using $m_c = 1.4$ GeV one arrives at $\beta = 2.9 \pm 0.4$ GeV^{-1} §, where the errors reflect the experimental errors.

On the other hand one can also use the ratio of partial decay width in D^{*0} system $\Gamma(D^{*0} \to D^0\gamma) : \Gamma(D^{*0} \to D^0\pi^0) = (38.1 \pm 2.9) : (61.9 \pm 2.9)$, where the experimental errors are considerably smaller than in the previous case. At tree level one has

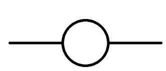
$$\frac{\Gamma(D^{*0} \to D^0\gamma)}{\Gamma(D^{*0} \to D^0\pi^0)} = \frac{e^2}{12\pi}\frac{k_\gamma^3}{k_\pi^3}\frac{12\pi f^2}{g^2}\left(\frac{2}{3}\beta + \frac{2}{3}\frac{1}{m_c}\right)^2, \tag{2.55}$$

with k_γ,k_π the momenta of the outgoing photon and pion respectively. Using $m_c = 1.4$ GeV, $g = 0.59$, $f = f_\pi = 130.7$ MeV one arrives at $\beta = 2.3 \pm 0.2$ GeV^{-1}, ¶ where the quoted errors again reflect only the errors on the input parameters coming from experiment. The β coupling coming from from D^{*+} (2.54) and D^{*0} (2.55) are in fair agreement, but not equal. This signals that other contributions coming from chiral loops and higher order terms that would alter our determination of β might be important. Since the contribution of chiral loops to $\Gamma(D^{*+} \to D^+\gamma)$ are approximately 50%, while for $D^{*0} \to D^0\gamma$ they are about 20% [?], we use in our numerical calculations the value of $\beta = 2.3$ GeV^{-1} obtained from $\Gamma(D^{*0} \to D^0\gamma)$.

Wave function renormalization

The values of couplings at the 1-loop depend on the regularization and renormalization prescriptions. Values for two renormalization prescriptions will be given, for the $\overline{\text{MS}}$ scheme and for the renormalization prescription $\bar{\Delta} = 2/\epsilon - \gamma + \ln(4\pi) + 1 \to 0$ as used by Gasser and Leutwyler in their analysis [?]. We will first discuss the calculation of wave function renormalization factors and then move on to the values of couplings α, g, f at one loop.

The wave function renormalization factors are defined as follows. We discuss first the case of light pseudoscalars. Let us define the sum of all one particle irreducible diagrams (1PI)‖ contributing to the light pseudoscalar propagator

$-iM^2(p^2) =$ 

§There is also a solution of (2.54) $\beta = 0.09 \pm 04$ GeV^{-1} that, however, does not agree with the determination of β from the D^{*0} decay.

¶The other solution is $\beta = -3.6 \pm 0.2$ GeV^{-1} that does not agree with D^{*+} data.

‖1PI diagram is a diagram that does not become disconnected, if any of the internal lines is cut.

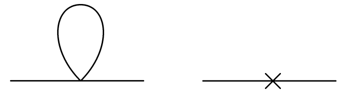


Figure 2.4: The $\mathcal{O}(p^4)$ 1PI amputated diagrams, that contribute to the light pseudoscalar wave function renormalization. The cross denotes counterterm contributions.

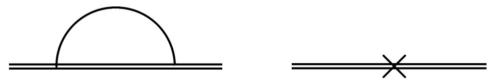
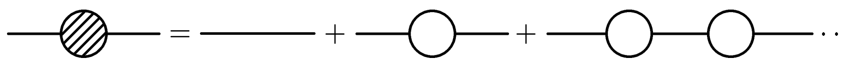


Figure 2.5: The $\mathcal{O}(p^3)$ contributions to the heavy meson wave function renormalization. The cross denotes counterterm contributions.

where the amputated 1PI on the right-hand side is understood, while p is the momentum running through the diagram. An infinite sum of 1PI diagrams represents the full light pseudoscalar propagator



The full light pseudoscalar propagator is thus a geometric sum of 1PI amputated diagrams and the intermediate bare propagators. After resummation this gives for the full propagator

Contributions to the wave function renormalizations for the light pseudoscalars K, π at the $\mathcal{O}(p^4)$ order in the chiral counting are shown on Figure 2.4. Explicitly they are

$$\begin{aligned} Z_K = 1 - \frac{1}{16\pi^2 f^2}\left[A_0(m_K^2) + \frac{1}{2}A_0(m_\pi) + \frac{1}{2}A_0(m_\eta^2)\right] \\ - 8L_5\frac{4\mu_0}{f^2}(\hat{m} + m_s) - 16L_4\frac{4\mu_0}{f^2}(m_u + m_d + m_s), \end{aligned} \tag{2.56}$$

$$Z_\pi = 1 - \frac{1}{16\pi^2 f^2}\left[\frac{2}{3}A_0(m_K^2) + \frac{4}{3}A_0(m_\pi^2)\right] - 8L_5\frac{4\mu_0}{f^2}\ 2\hat{m} - 16L_4\frac{4\mu_0}{f^2}(m_u + m_d + m_s), \tag{2.57}$$

with $A_0(m^2)$ function defined in appendix **??**, Eq. (**??**), while $m_{u,d,s}$ are quark masses with $\hat{m} = \frac{1}{2}(m_u = m_d)$. The counterterm contributions $L_{4,5}$ come from the insertions of terms given in Eq. (2.29).

The one chiral loop contributions to the heavy meson wave function renormalizations are shown on Figure 2.5. Explicit expressions for the heavy pseudoscalar and vector mesons D_a, D_a^*, containing $\bar{q}_a$ light valence antiquark, are

$$Z_{D_a} = 1 - \frac{3g^2}{16\pi^2 f^2}\sum_i (t^i)_{ab}(t^{i\dagger})_{ba} B'(\Delta_{D_b^* D_a}, m_{P_i}) + k_1 m_a + k_2(m_u + m_d + m_s), \tag{2.58}$$

$$\begin{aligned} Z_{D_a^*} = 1 - \frac{2g^2}{16\pi^2 f^2}\sum_i (t^i)_{ab}(t^{i\dagger})_{ba}\left[\tfrac{1}{2}B'(\Delta_{D_b D_a^*}, m_{P_i}) + B'(\Delta_{D_b^* D_a^*}, m_{P_i})\right] \\ + k_1 m_a + k_2(m_u + m_d + m_s), \end{aligned} \tag{2.59}$$

where the summation over a is suspended, while $\Delta_{H_1H_2} = m_{H_1} - m_{H_2}$, $m_{u,d,s}$ are the quark masses, and $B'(\Delta, m) = \frac{\partial}{\partial\Delta}\bar{B}_{00}(m,\Delta)$, where $\bar{B}_{00}(m,\Delta)$ can be found in appendix **??**, Eq. (**??**). The $SU(3)$ matrices $t^i$are defined through $\Pi = P^i t^i$, Eq. (2.27). In the heavy quark limit $m_Q \to \infty$ we have $\Delta \to 0$ and the two renormalizations are equal.

The decay constants

The light pseudoscalar decay constants receive contributions at one chiral loop level from diagrams on Fig. 2.6. For π, K they are

$$\begin{aligned} f_K = f\Big(1 + \frac{1}{16\pi^2 f^2}[A_0(m_\eta^2) + 2A_0(m_K^2) + A_0(m_\pi^2)] + 8L_5\frac{4\mu_0}{f^2}(m_s + \hat{m}) \\ + 16L_4\frac{4\mu_0}{f^2}(m_u + m_d + m_s)\Big)\sqrt{Z_K}, \end{aligned} \tag{2.60}$$

$$f_\pi = f\Big(1 + \frac{1}{16\pi^2 f^2}\frac{4}{3}[A_0(m_K^2) + 2A_0(m_\pi^2)] + 8L_5\frac{4\mu_0}{f^2}2\hat{m} + 16L_4\frac{4\mu_0}{f^2}(m_u + m_d + m_s)\Big)\sqrt{Z_\pi}, \tag{2.61}$$

where the wave-function renormalizations are given in (2.56), (2.57). We use the expression for the pion decay constant, together with the experimental value $f_\pi = 130.7 \pm 0.4$ MeV [**?**], from which at one loop $f = 0.12 \pm 0.01$ GeV [**?**] both in $\overline{\text{MS}}$ and Gasser-Leutwyler prescriptions. The error is due to the poorly known L_4 counterterm, that will be discussed latter on in this section in somewhat more detail (cf. Eqs. (**??**), (**??**)).

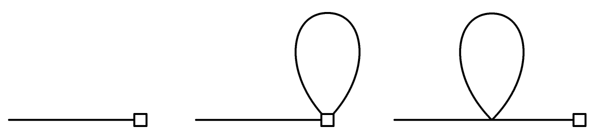


Figure 2.6: Leading and one loop chiral corrections to f_π, f_K decay constants. The light pseudoscalars are represented by solid line, square denotes weak current insertion.

The decay constants f_D, f_{D_s} receive contributions from diagrams depicted on Fig. 2.7. For $\bar{\Delta} \to 0$ these have been calculated in [**?**, **?**], while the leading logs have been obtained already in [**?**, **?**]. Taking into account the counterterms, the 1-loop expressions are

$$\begin{aligned} f_D = \frac{\alpha}{\sqrt{m_D}}\Big[1 - \frac{3g^2}{32\pi^2 f^2}\Big(\frac{3}{2}B'(\Delta_{D^*D}, m_\pi) + B'(\Delta_{D_s^*D}, m_K) + \frac{1}{6}B'(\Delta_{D^*D}, m_\eta)\Big) \\ + \frac{1}{32\pi^2 f^2}\Big(\frac{3}{2}A_0(m_\pi^2) + A_0(m_K^2) + \frac{1}{6}A_0(m_\eta^2)\Big) \\ + \big(\tfrac{1}{2}k_1 + \varkappa_1\big)\,\hat{m} + \big(\tfrac{1}{2}k_2 + \varkappa_2\big)(2\hat{m} + m_s)\Big], \end{aligned} \tag{2.62a}$$

$$\begin{aligned} f_{D_s} = \frac{\alpha}{\sqrt{m_{D_s}}}\Big[1 - \frac{3g^2}{32\pi^2 f^2}\Big(2B'(\Delta_{D^*D_s}, m_K) + \frac{2}{3}B'(\Delta_{D_s^*D_s}, m_\eta)\Big) \\ + \frac{1}{32\pi^2 f^2}\Big(2A_0(m_K^2) + \frac{2}{3}A_0(m_\eta^2)\Big) + \big(\tfrac{1}{2}k_1 + \varkappa_1\big)\,m_s + \big(\tfrac{1}{2}k_2 + \varkappa_2\big)(2\hat{m} + m_s)\Big], \end{aligned} \tag{2.62b}$$

where $\Delta_{H_1H_2} = m_{H_1} - m_{H_2}$, $\hat{m} = \frac{1}{2}(m_u + m_d)$ with $m_{u,d,s}$ the quark masses, while $B'(\Delta, m) = \frac{\partial}{\partial\Delta}\bar{B}_{00}(m,\Delta)$, and $A_0(m^2)$, $\bar{B}_{00}(m,\Delta)$ can be found in appendix **??**, Eqs. (**??**), (**??**). The formulas

(2.62) are valid at the leading order in $1/m_Q$ [?, ?]. Evaluating expression for f_{D_s} (2.62b) using the tree level values for g, f and the experimental value of f_{D_s} one arrives at $\alpha^{\overline{\mathrm{MS}}} = 0.21 \pm 0.05$ GeV$^{3/2}$ in $\overline{\mathrm{MS}}$ scheme, and $\alpha^{\mathrm{GL}} = 0.24 \pm 0.05$ GeV$^{3/2}$ in the Gasser-Leutwyler prescription. The error is equally distributed between experimental errors in f_{D_s}, experimental error in g^{Tree} and variation of unknown counterterms as described below (cf. Eqs. (**??**), (**??**) and the text below them). The variation of the counterterms introduces relatively large error as they are proportional to m_K^2/f^2. Estimated error is only approximate also because $1/m_D$ correction have been neglected.

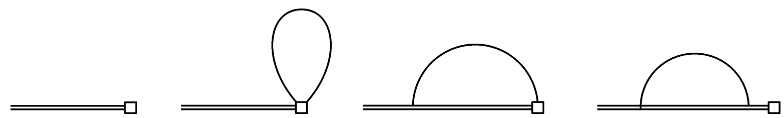


Figure 2.7: The leading contribution and the one loop chiral corrections to f_D, f_{D_s} decay constants. The double line represents heavy meson, solid line the light pseudoscalar mesons, while the square denotes the weak current insertion. The diagram with one light pseudoscalar attached to the weak current (second from right) vanishes.

One loop corrections to the $D^*D\pi$ coupling

The contributions to the $D^*D\pi$ coupling at one chiral loop are shown on Fig. 2.8. They give

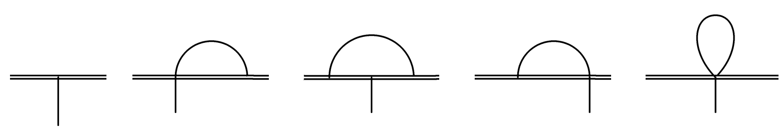


Figure 2.8: Leading and one loop chiral corrections to $g_{D^*D\pi}$ coupling. The double line represents heavy meson, solid line the light pseudoscalar mesons. The second and fourth diagrams vanish.

Comparing with the experimental value [?, ?] $g_{D^*D\pi} = g^{\mathrm{Tree}} = 0.59 \pm 0.08$, we arrive at $g^{\overline{\mathrm{MS}}} = 0.41 \pm 0.10$, $g^{\mathrm{GL}} = 0.49 \pm 0.10$, where the values of $L_{4,5}$ have been used as given in Table 2.1, while other counterterms have been set to zero as discussed above. The error in $g^{\overline{\mathrm{MS}}}$, g^{GL} is experimental and from the uncertainty on the L_4 counterterm (this is proportional to m_K^2, see (2.57)). The error from the other unknown counterterms can be estimated to be of the same order.

Counterterms

The values of the counterterms L_4 and L_5 in (2.29) are taken from [?] and are scaled to $\mu = 1$ GeV using

There are no experimental data regarding the sizes of the $k_{1,2}$ and $\varkappa_{1,2}$ counterterms in (2.45) and (2.46). From large N_c considerations we can conclude that k_2 and $\varkappa_2$ are $1/N_c$ suppressed, i.e., the following relations are expected $k_1 > k_2$, $\varkappa_1 > \varkappa_2$. In the numerical estimates we then set

$k_2 = \varkappa_2 = 0$. The approximate size of k_1 is determined by observing that L_5 term in (2.29) and k_1 term in (2.45) have similar structure compared to the kinetic term in (2.28) and (2.43) respectively. It is thus reasonable to expect that roughly $|k_1| \sim L_5 4\mu_0 8/f^2$. Similar reasoning applies for $\varkappa_1$, so that in the numerical evaluation we vary k_1 and $\varkappa_1$ in the range $-32L_5\mu_0/f^2 < k_1, \varkappa_1 < 32L_5\mu_0/f^2$.

–	Tree	1-loop $\overline{\text{MS}}$	1-loop GL
α [GeV$^{3/2}$]	0.38 ± 0.04	0.21 ± 0.05	0.24 ± 0.05
g	0.59 ± 0.08	0.41 ± 0.10	0.49 ± 0.10
f [MeV]	130.7 ± 0.04	120 ± 10	120 ± 10
β [GeV^{-1}]	2.3 ± 0.2	–	–
L_4 [$\times 10^{-3}$]	–	-0.9 ± 0.5	-0.5 ± 0.5
L_5 [$\times 10^{-3}$]	–	-0.6 ± 0.5	0.6 ± 0.5

Table 2.1: Values of coupling constants used further on in the analysis. The $\overline{\text{MS}}$ and GL renormalization prescriptions correspond to letting respectively $\bar{\Delta} \to 1$ and $\bar{\Delta} \to 0$ in the loop integrals. The values of other $\mathcal{O}(p^4)$ terms are either set to zero or varied in the ranges discussed in text.

At the end let us summarize the approximations that were made in obtaining the 1-loop values of couplings g, α, f given in Table 2.1

- The $1/m_c$ contributions have been neglected. These are expected to be more important in the determination of α, as in it contributions of order m_K/m_D might arise. The $1/m_c$ corrections are expected to be less important in the determination of g, where they are proportional to m_π.

- In the determination of g a number of unknown counterterms have been set to zero. Except for the $\tilde{\varkappa}_3$ they are proportional to m_π^2 which justifies this procedure. The $\tilde{\varkappa}_3$ contribution is proportional to m_K^2, while $\tilde{\varkappa}_3$ itself is of order $1/N_c$ and is expected to be suppressed (**??**). The situation is very similar to the case of L_4 contribution, which is proportional to m_K^2, while L_4 is $1/N_c$ suppressed. Note also that the change of scale and/or renormalization prescription can invalidate the $1/N_c$ argument as can be seen from the relatively large value of $L_4^{\overline{\text{MS}}}$.

- The uncertainties connected with the couplings in the heavy meson sector do not influence determination of f at one loop.

Chapter 3

One loop scalar and tensor functions

In the following we shall present the calculation of the dimensionally regularized one loop scalar functions with one heavy meson propagator, that has been published in [?]. These will be needed in the evaluation of radiative D meson decays discussed in chapter ??. The expressions for the dimensionally regularized one loop scalar functions within full theory have been know for a long time [?]. The full expressions for scalar functions with one heavy meson propagator, on the other hand, have not been calculated until recently [?, ?, ?, ?, ?, ?].

The one loop calculations within the heavy quark effective theory are considerably simplified if the light-quark masses are neglected. Very common in the heavy hadron chiral perturbation theory (HHχPT) is a similar approximation, with the finite contributions omitted, while only the leading logs are retained [?, ?]. To go beyond the leading log approximation and/or take into account the counterterms appearing at the next order in the chiral expansion, the general solutions for the one loop scalar functions need to be considered. In the context of the HHχPT a general solution for the one loop scalar two-point function with one heavy quark propagator has been found in [?, ?]. We extend this calculation and find solutions for the scalar three-point and four-point functions with one heavy quark propagator.

The vector and tensor one loop functions can then be expressed in terms of the scalar one loop functions using the algebraic reduction [?]. Also, the one loop scalar functions with two or more heavy quark propagators can be expressed in terms of the one loop scalar functions with just one heavy quark propagator. For the case of the equal heavy-quark velocities this can be accomplished using the relation

$$\frac{1}{v\cdot q-\Delta}\frac{1}{v\cdot q-\Delta'}=\frac{1}{\Delta-\Delta'}\left(\frac{1}{v\cdot q-\Delta}-\frac{1}{v\cdot q-\Delta'}\right). \tag{3.1}$$

For unequal heavy quark velocities techniques developed in [?] can be used.

The scalar one-loop functions with heavy quark propagators can be derived also directly from the scalar functions of the full theory by using the threshold expansion [?] (see also appendix B of [?]). This technique has recently been used for the calculation of the scalar and tensor three-point functions with one and two heavy quark propagators [?, ?]. We will not, however, follow the approach of Bouzas et al. [?, ?, ?] but rather do the calculation from scratch.

This chapter is organized as follows: first we will introduce the notation for scalar and tensor functions that will be used further on. Then we shall proceed to the evaluation of scalar functions. At the beginning we will make some general remarks and list useful relations that will be used further on in the calculation. Then we will review the calculation of one and two point functions. We will continue with the calculation of the three-point and four-point functions in the final sections.

3.1 Notational conventions for loop integrals

In this section we list the definitions of the dimensionally regularized integrals commonly encountered in the evaluation of the χPT and HHχPT one-loop diagrams. The integrals containing a heavy quark propagator are

$$-\frac{1}{16\pi^2}\bar{A}_0(m) = \frac{i\mu^\epsilon}{(2\pi)^n}\int d^n q \frac{1}{(v\cdot q-\Delta+i\delta)} = 0, \tag{3.2}$$

$$-\frac{1}{16\pi^2}\bar{B}_{\{0,\mu,\mu\nu\}}(m,\Delta) = \frac{i\mu^\epsilon}{(2\pi)^n}\int d^n q \frac{\{1,q_\mu,q_\mu q_\nu\}}{(v\cdot q-\Delta+i\delta)(q^2-m^2+i\delta)}, \tag{3.3}$$

$$\begin{aligned}-\frac{1}{16\pi^2}\bar{C}_{\{0,\mu,\mu\nu\}}&(p,m_1,m_2,\Delta) = \\ &\frac{i\mu^\epsilon}{(2\pi)^n}\int d^n q \frac{\{1,q_\mu,q_\mu q_\nu\}}{(v\cdot q-\Delta)(q^2-m_1^2)((q+p)^2-m_2^2)},\end{aligned} \tag{3.4}$$

$$\begin{aligned}-\frac{1}{16\pi^2}\bar{D}_{\{0,\mu,\mu\nu\}}&(p_1,p_2,m_1,m_2,m_3,\Delta) = \\ &\frac{i\mu^\epsilon}{(2\pi)^n}\int \frac{d^n q\qquad\{1,q_\mu,q_\mu q_\nu\}}{(v\cdot q-\Delta)(q^2-m_1^2)((q+p_1)^2-m_2^2)((q+p_2)^2-m_3^2)},\end{aligned} \tag{3.5}$$

where $n=4-\epsilon$. The dependence of scalar and tensor functions on v^μ is not shown explicitly and also in Eqs. (3.4), (3.5) the $i\delta$ prescription is not shown. The scalar integrals $\bar{B}_0(m,\Delta)$, $\bar{C}_0(p,m_1,m_2,\Delta)$, $\bar{D}_0(p_1,p_2,m_1,m_2,m_3,\Delta)$ have been calculated in [?]. We use the expressions of Ref. [?] in the numerical evaluation of the scalar integrals $\bar{B}_0$, $\bar{C}_0$, $\bar{D}_0$. The tensor integrals can be expressed in terms of Lorentz-covariant tensors. The notation we use for the tensor functions resembles closely the notation used in Ref. [?] for the Veltman-Passarino functions [?]

$$\bar{B}_\mu(m,\Delta) = v_\mu \bar{B}_1, \tag{3.6}$$
$$\bar{B}_{\mu\nu}(m,\Delta) = \eta_{\mu\nu}\bar{B}_{00} + v_\mu v_\nu \bar{B}_{11}, \tag{3.7}$$
$$\bar{C}_\mu(p,m_1,m_2,\Delta) = v_\mu\bar{C}_1 + p_\mu\bar{C}_2, \tag{3.8}$$
$$\bar{C}_{\mu\nu}(p,m_1,m_2,\Delta) = \eta_{\mu\nu}\bar{C}_{00} + (v_\mu p_\nu + p_\mu v_\nu)\bar{C}_{12} + v_\mu v_\nu\bar{C}_{11} + p_\mu p_\nu\bar{C}_{22}, \tag{3.9}$$
$$\bar{D}_\mu(p_1,p_2,m_1,m_2,m_3,\Delta) = v_\mu\bar{D}_1 + p_{1\mu}\bar{D}_2 + p_{2\mu}\bar{D}_3, \tag{3.10}$$
$$\begin{aligned}\bar{D}_{\mu\nu}(p_1,p_2,m_1,m_2,m_3\Delta) = {}&\eta_{\mu\nu}\bar{D}_{00} + v_\mu v_\nu\bar{D}_{11} + (v_\mu p_{1\nu} + p_{1\mu}v_\nu)\bar{D}_{12} + p_{1\mu}p_{2\nu}\bar{D}_{23} \\ &+ p_{1\mu}p_{1\nu}\bar{D}_{22} + (v_\mu p_{2\nu} + p_{2\mu}v_\nu)\bar{D}_{13} + p_{2\mu}p_{2\nu}\bar{D}_{33}.\end{aligned} \tag{3.11}$$

The tensor functions are calculated using the algebraic reduction [?], i.e., the tensor functions (3.6)-(3.11) are multiplied by the four-momenta $v^\mu, p^\mu, \ldots$ or contracted using $\eta^{\mu\nu}$. Then the identities such as $v\cdot q = v\cdot q - \Delta + \Delta$ and/or $q\cdot p = \frac{1}{2}((q+p)^2 - m^2 - (q^2-m^2))$ are used to reduce tensor integrals to a sum of scalar integrals. The result of this procedure has been given explicitly in [?] for the case of two point functions $\bar{B}_{\{\mu,\mu\nu\}}$ *. For the case of the three and four-point functions $\bar{C}_{\{\mu,\mu\nu\}}$, $\bar{D}_{\{\mu,\mu\nu\}}$ we do not write out explicitly the analytic results of algebraic reductions as the expressions are relatively cumbersome. For instance in the case of $\bar{D}_{\mu\nu}$ the final expression involves the inverse of a 7×7 matrix that corresponds to seven functions $\bar{D}_{00}\ldots\bar{D}_{33}$ appearing in the expression of the four-point tensor function (3.11). Note as well that in this particular case there are ten possible relations between $\bar{D}_{00}\ldots\bar{D}_{33}$ and the scalar functions $\bar{B}_0$, $\bar{C}_0$, $\bar{D}_0$ that one gets from algebraic reductions (three equations from each multiplication by v^μ, p_1^μ, p_2^μ plus one relation from

*Note that different notation is used in Ref. [?], with $\bar{B}_0(m,\Delta) = -I_2(m,\Delta)/\Delta$, $\bar{B}_1(m,\Delta) = -I_2(m,\Delta) - I_1(m)$, $\bar{B}_{00}(m,\Delta) = -\Delta J_1(m,\Delta)$, $\bar{B}_{11}(m,\Delta) = -\Delta J_2(m,\Delta)$.

contraction by $\eta^{\mu\nu}$). Obviously not all ten equations can be linearly independent. Using different sets of seven independent equations have to lead to the same results for $\bar{D}_{00}\ldots\bar{D}_{33}$ coefficient functions. This fact can then be used as a very useful check in the numerical implementation.

The loophole of the aforementioned procedure is, if the set of equations provided by the algebraic reduction is not invertible. This happens for instance in the calculation of $D^0 \to l^+l^-\gamma$ (see section **??**). Namely, for $p_1 = p$ and $p_2 = p + k$ appearing in the calculation of $C_0^{4\text{-}4}$ (with p the four-momentum of the lepton pair and k the photon momentum, see section **??** or appendix **??**, Eqs. (**??**), (**??**)) only six out of ten relations following from algebraic reduction are linearly independent. This problem is connected to the special kinematics of $D^0 \to l^+l^-\gamma$ decays and has been circumvented by first calculating the tensor four-point functions with the prescription $k \to k + \epsilon a$, where a is some arbitrary four-momentum, and then taking the limit $\epsilon \to 0$ numerically. Similarly, in the calculation of $C_0^{4\text{-}5}$, where $p_1 = k$, $p_2 = k + p$, see Eqs. (**??**), (**??**), the prescription $p \to p + \epsilon a$ has been used. Because $\bar{D}_{00}\ldots\bar{D}_{33}$ are continuous functions of p_1 and p_2, the outlined limiting procedure leads to an unambiguous result. This has been also checked numerically.

To make the listing of notational conventions self-contained we give in the following also the notation for the Veltman-Passarino functions employed by the LoopTools package [**?**], that has been used for their numerical evaluation. A general integral is

$$-\frac{1}{16\pi^2}T^N_{\mu_1\ldots\mu_P} = \frac{i\mu^\epsilon}{(2\pi)^n}\int\frac{d^nq \qquad\qquad q_{\mu_1}\cdots q_{\mu_P}}{(q^2-m_1^2)((q+p_1)^2-m_2^2)\cdots((q+p_{N-1})^2-m_N^2)}, \tag{3.12}$$

with two-point functions T^2 generally denoted by the letter B, the three-point functions T^3 by the letter C and the four-point functions T^4 by the letter D. Thus, e.g., $B_0(p^2, m_1^2, m_2^2)$ and $C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_1^2, m_2^2)$ are two-point and three-point scalar functions respectively. The decomposition of the tensor integrals in terms of the Lorentz-covariant tensors reads explicitly

$$B_\mu = p_{1\mu}B_1, \tag{3.13}$$
$$B_{\mu\nu} = \eta_{\mu\nu}B_{00} + p_{1\mu}p_{1\nu}B_{11}, \tag{3.14}$$
$$C_\mu = p_{1\mu}C_1 + p_{2\mu}C_2 = \sum_{i=1}^{2}p_{i\mu}C_i, \tag{3.15}$$
$$C_{\mu\nu} = \eta_{\mu\nu}C_{00} + \sum_{i,j=1}^{2}p_{i\mu}p_{j\nu}C_{ij}, \tag{3.16}$$
$$C_{\mu\nu\rho} = \sum_{i=1}^{2}(\eta_{\mu\nu}p_{i\rho} + \eta_{\nu\rho}p_{i\mu} + \eta_{\mu\rho}p_{i\nu})C_{00i} + \sum_{i,j,l=1}^{2}p_{i\mu}p_{j\nu}p_{l\rho}C_{ijl}. \tag{3.17}$$

Note that the tensor-coefficient functions are totally symmetric in their indices.

3.2 General remarks and useful relations

Let us now turn to the evaluation of the one loop scalar functions, with one heavy quark propagator (3.2)-(3.5). Before we start with the actual calculation, let us, however, first list some useful relations and the conventions that are going to be used further on. The greater part of this section is a review of the relations and the conventions used in [**?**] with certain modifications. The major difference between the conventional one loop scalar functions and the one loop scalar functions with one heavy quark propagator is the appearance of the propagator linear in the integration variable q. Therefore, a modified version of the standard Feynman parameterization is used

$$\begin{aligned}\frac{1}{(\prod_{i=1}^N A_i)B} =& N! \int_0^\infty 2d\lambda \int \prod_{i=1}^N du_i \, \frac{\delta(1-\sum_i u_i)\ \prod_i \Theta(u_i)}{[\sum_{i=1}^N A_i u_i + 2B\lambda]^{N+1}} \\ =& N! \, \frac{\int_0^\infty 2d\lambda \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{N-2}} dx_{N-1}}{[A_1(1-x_1) + \sum_{j=2}^{N-1} A_j(x_{j-1}-x_j) + A_N x_{N-1} + 2B\lambda]^{N+1}},\end{aligned} \tag{3.18}$$

where $\Theta(u)$ is the Heaviside function $\Theta(u) = 1$ for $u > 0$ and zero otherwise. In the calculation A_i are going to be "full" (inverse) propagators $((q+p_i)^2 - m_i^2 + i\delta)$ and B the heavy quark propagator $(v{\cdot}q - \Delta + i\delta)$. Note also, that the leading power of q^2 in the denominator has increased from the left-hand side's $(q^2)^{N+1/2}$ to the right-hand side's $(q^2)^{N+1}$. The integration over q has been made more convergent, but then another integration over infinite range (integration over λ) has been introduced through the parameterization.

A very useful identity used in the calculation is

$$\begin{aligned}\frac{1}{[(q+p_1)^2 - m_1^2 + i\delta][(q+p_2)^2 - m_2^2 + i\delta]} =& \frac{\alpha}{[(q+p_1)^2 - m_1^2 + i\delta][(q+l)^2 - M^2 + i\delta]} \\ &+ \frac{1-\alpha}{[(q+p_2)^2 - m_2^2 + i\delta][(q+l)^2 - M^2 + i\delta]},\end{aligned} \tag{3.19}$$

where α is an arbitrary parameter and

$$l = p_1 + \alpha(p_2 - p_1), \tag{3.20}$$

$$M^2 = (1-\alpha)m_1^2 + \alpha m_2^2 - \alpha(1-\alpha)(p_2-p_1)^2. \tag{3.21}$$

The parameter α can then be chosen at will. It is useful to keep it real, though. Then there are no ambiguities connected with the shift of the integration variable q, that is performed, as usual, before the Wick rotation. For instance α can be chosen such that $M^2 = 0$. If $(p_2 - p_1)^2 \leqslant (m_1 - m_2)^2$ or $(p_2 - p_1)^2 \geqslant (m_1 + m_2)^2$, then α is real. If one of the masses is made to be zero, the integration is simplified considerably (as will be seen in the calculation of the four-point function (3.66)). The other option used below is to set α such that $l^2 = 0$. This can be done for real α if (but not only if) one of p_1, p_2 or $p_1 \pm p_2$ is timelike. This shows, that in general product of propagators at least one internal or one external mass can always be set to zero, even with α restricted to be real.

In doing the integrals the following procedure proves to be very useful. Consider

$$\int_0^\infty 2d\lambda \int_0^1 dx \frac{1}{[ax^2 + b\lambda^2 + cx\lambda + \cdots]}. \tag{3.22}$$

The integration over x can be simplified by the change of the integration variables $\lambda = \lambda' + \beta x$, where β is chosen such, that the coefficient in front of x^2 vanishes, i.e., β has to solve the equation $b\beta^2 + c\beta + a = 0$. Then the integrand is linear in x, so the integration over x is trivial. The integration bounds are

$$\begin{aligned}\int_0^\infty 2d\lambda \int_0^1 dx \cdots =& \int_0^1 dx \int_{-\beta x}^\infty 2d\lambda' \cdots = \\ =& \int_0^\infty 2d\lambda \int_0^1 dx \cdots + \int_{-\beta}^0 2d\lambda \int_{-\lambda/\beta}^1 dx \cdots.\end{aligned} \tag{3.23}$$

As the results of the integration, the functions such as logarithms, dilogarithms (Spence functions) and hypergeometric functions will appear. Since the arguments of the functions will in general lie in the complex plane it is necessary to discuss the conventions used. The convention used for the logarithms is that they have a cut along the negative real axis. For x exactly negative real we use the prescription $\ln(x) \to \ln(x + i\epsilon)$, where $\epsilon > 0$ is a positive infinitesimal parameter. In other words, $\ln(x) = \ln|x| + i\pi$ for x negative real[†]. In particular $\ln(-1)$ is defined to be $\ln(-1) = i\pi$. Of course this choice is completely arbitrary and at the end of the calculation one has to check that results are independent of this choice. Using this definition for the logs of the negative real arguments the logarithm of an inverse is

$$\ln\left(\frac{1}{x}\right) = -\ln(x) + 2\pi i \Re^{(-)}(x), \tag{3.24}$$

with

$$\Re^{(-)}(x) = \begin{cases} 1\ ; & x \text{ on negative real axis,} \\ 0\ ; & \text{otherwise.} \end{cases} \tag{3.25}$$

Note that the change from the usual rule for the logarithm of an inverse is just on the negative real axis. For the arguments away from the negative real axis the function $\Re^{(-)}(x)$ is exactly zero and everything is as usual. The logarithm of a product is

$$\ln(ab) = \ln(a) + \ln(b) + \eta(a, b), \tag{3.26}$$

where η function is [‡]

$$\eta(a,b) = \begin{cases} 2\pi i\big\{\Theta(-\Im(a))\Theta(-\Im(b))\Theta(\Im(ab)) \\ \qquad - \Theta(\Im(a))\Theta(\Im(b))\Theta(-\Im(ab))\big\}\ ; & a \text{ and } b \textit{ not} \text{ negative real,} \\ -2\pi i\big\{\Theta(\Im(a)) + \Theta(\Im(b))\big\}\ ; & \text{either } a \text{ or } b \text{ negative real,} \\ -2\pi i\ ; & a \text{ and } b \text{ negative real.} \end{cases} \tag{3.27}$$

The normal rule for the logarithm of a product applies for these important cases

$$\begin{aligned} \ln(ab) &= \ln(a) + \ln(b);\ \Im(a) \text{ and } \Im(b) \text{ have the opposite sign,} \\ \ln(a/b) &= \ln(a) - \ln(b);\ \Im(a) \text{ and } \Im(b) \text{ have the same sign,} \end{aligned} \tag{3.28}$$

with a, b not negative real.

A very illuminative example of what kind of problems are encountered when doing the integrals of functions with branch cuts in the complex plane is the following simple calculation taken from [**?**]. Consider for instance

$$I(a,b) = \int_0^1 \frac{dx}{ax+b}, \tag{3.29}$$

[†]Note that this prescription does not change the calculation of the logarithms away from the negative real axis. In particular it does not change the value of a logarithm with an argument that already has an infinitesimal but nonzero imaginary part. For more discussion on this point see text after Eq. (3.106).

[‡]Note, that in comparison with [**?**], the η function has been extended also to the negative real arguments (cf. discussion after Eq. (3.75) and Eq. (3.88)). For arguments away from the negative real axis (also if by an infinitesimal amount) it is the same as in [**?**].

where a and b are arbitrary complex numbers. The indefinite integral would be $a^{-1}\ln(ax+b)$, but then one has to take care whether for $x \in [0,1]$ the argument of the logarithm crosses the negative real axis or not. It is easier to first divide out a

$$I(a,b) = \frac{1}{a}\int_0^1 \frac{dx}{x+b/a} = \frac{1}{a}\ln\left(x+\frac{b}{a}\right)\Big|_0^1. \tag{3.30}$$

Then the argument of the logarithm does not cross the negative real axis as it has the same imaginary part regardless of the value of (real) x. So

$$I(a,b) = \frac{1}{a}\left[\ln\left(1+\frac{b}{a}\right) - \ln\frac{b}{a}\right]. \tag{3.31}$$

Since the arguments of the logarithms have the imaginary parts of the same sign, the usual rule (3.28) applies and one can write

$$I(a,b) = \frac{1}{a}\ln\frac{a+b}{a}, \tag{3.32}$$

which is actually the standard result. The problem with the careless derivation would be, that it would give $a^{-1}(\ln(a+b)-\ln a)$, which is not correct for all choices of a and b. For instance, for $a=-1-i\varepsilon$ infinitesimally below the negative real axis and $b=-1+2i\varepsilon$ infinitesimally above the negative real axis, the integration path lies infinitesimally close to the negative real axis with the starting point below and ending point above the negative real axis. Since the integration path does not cross the pole of the integrand in (3.29), the result of integration should be almost real. The naive result $a^{-1}(\ln(a+b)-\ln a)$, however, gives incorrectly $-\ln 2 - 2\pi i$, while the use of Eq. (3.32) leads to the correct result $-\ln 2$.

3.3 Dilogarithm and hypergeometric function

In this section we list some properties of the dilogarithm and the hypergeometric function used in the rest of this chapter (for other properties consult, e.g., [?, ?]).

The dilogarithm or Spence function is defined as

$$\mathrm{Li}(x) = -\int_0^1 dt\,\frac{\ln(1-xt)}{t}. \tag{3.33}$$

The cut for the logarithm along the negative real axis translates into the cut for the dilogarithm along the positive real axis for $x>1$. For x on the positive real axis, $x>1$, dilogarithm is calculated using the following prescription $\mathrm{Li}(x) \to \mathrm{Li}(x-i\epsilon)$. Note as well that $\mathrm{Li}(0)=0$.

Useful identities valid also for the complex arguments (not equal to zero) are [?]

$$\mathrm{Li}(x) = -\,\mathrm{Li}\left(\frac{1}{x}\right) - \frac{1}{6}\pi^2 - \frac{1}{2}\big[\ln(-x) - 2\pi i\xi(x)\big]^2, \tag{3.34a}$$

$$\mathrm{Li}(x) = -\,\mathrm{Li}(1-x) + \frac{1}{6}\pi^2 - \ln(x)\ln(1-x), \tag{3.34b}$$

where

$$\xi(x) = \begin{cases} 1\,; & x\in(0,1), \\ 0\,; & \text{otherwise}. \end{cases} \tag{3.35}$$

The hypergeometric function for complex argument $|z| < 1$ is defined in terms of the series

$$ {}_2F_1(\alpha, \beta; \gamma; z) = 1 + \sum_{n=0}^{\infty} \frac{\alpha \dots (\alpha + n)\beta \dots (\beta + n)}{\gamma \dots (\gamma + n)(n + 1)!} z^{n+1}, \tag{3.36} $$

with γ not equal to zero or negative integer. Note that the series terminates if α or β are equal to negative integer or zero. If either of them is zero then

$$ {}_2F_1(0, \beta; \gamma; z) = {}_2F_1(\alpha, 0; \gamma; z) = 1. \tag{3.37} $$

For z outside the unit circle the values of the hypergeometric function can be obtained through analytic continuation. We make a cut in the z plane along the real axis from $z = 1$ to $z = \infty$. Then the series (3.36) will yield, in the cut plain, a single valued analytic continuation that can be obtained using the following identity (other similar transformation formulas can be found in, e.g., [?])

$$ \begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) =& \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-z)^{-\alpha} {}_2F_1(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; 1/z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (-z)^{-\beta} {}_2F_1(\beta, \beta + 1 - \gamma; \beta + 1 - \alpha; 1/z). \end{aligned} \tag{3.38} $$

The integral representations of the hypergeometric function include

$$ \int_z^{\infty} \frac{x^{\mu-1} dx}{(1 + \beta x)^{\nu}} = \frac{z^{\mu-\nu}}{\beta^{\nu}(\nu - \mu)} \; {}_2F_1(\nu, \nu - \mu; \nu - \mu + 1; -1/(\beta z)), \tag{3.39a} $$

$$ \int_0^{z} \frac{x^{\mu-1} dx}{(1 + \beta x)^{\nu}} = \frac{z^{\mu}}{\mu} \; {}_2F_1(\nu, \mu; 1 + \mu; -\beta z), \tag{3.39b} $$

where Eq. (3.39a) is valid for $\Re(\nu) > \Re(\mu)$, while Eq. (3.39b) is valid for the case, when $\arg(1+\beta z) < \pi$ and $\Re(\mu) > 0$.

3.4 One- and two-point functions

In this section we will concentrate on the calculation of the dimensionally regularized one-point and two-point functions in the heavy quark effective theory

$$ -\frac{1}{16\pi^2} \bar{A}_0(\Delta) = \frac{i\mu^{\epsilon}}{(2\pi)^n} \int d^n q \frac{1}{(v{\cdot}q - \Delta + i\delta)}, \tag{3.40} $$

$$ -\frac{1}{16\pi^2} \bar{B}_0(m, \Delta) = \frac{i\mu^{\epsilon}}{(2\pi)^n} \int d^n q \frac{1}{(v{\cdot}q - \Delta + i\delta)(q^2 - m^2 + i\delta)}, \tag{3.41} $$

where δ is a positive infinitesimal parameter, $n = 4-\epsilon$, while m and Δ are real. The general solution for the two-point function has been found by Stewart in Ref. [?] (see also [?] and references therein). In this section we will derive Stewart's result.

We start with the integral

$$ I_r = \frac{i\mu^{\epsilon}}{(2\pi)^n} \int d^n q \frac{1}{(v{\cdot}q - \Delta + i\delta)(q^2 - m^2 + i\delta)^r}. \tag{3.42} $$

Using the Feynman parameterization [?]

$$\frac{1}{a^r b^s} = 2^s \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^\infty d\lambda \frac{\lambda^{s-1}}{(a+2b\lambda)^{r+s}}, \tag{3.43}$$

we get

$$I_r = \mu^\epsilon \frac{(-1)^{-r}}{\Gamma(r)} \frac{2\Gamma(r+1-\frac{n}{2})}{(4\pi)^{\frac{n}{2}}} \int_0^\infty d\lambda (\lambda^2 + 2\lambda\Delta + m^2 - i\delta)^{\frac{n}{2}-r-1}. \tag{3.44}$$

The integral can be expressed in terms of the hypergeometric function. Introducing the new variable $\lambda' = \lambda + \Delta$ and then splitting the integration interval for negative Δ we get

$$\int_\Delta^\infty \frac{d\lambda'}{(\lambda'^2 - \Delta^2 + m^2 - i\delta)^N} = \int_{-|\Delta|}^{|\Delta|} \frac{d\lambda}{(\lambda^2 - \Delta^2 + m^2 - i\delta)^N} \Theta(-\Delta) + \int_{|\Delta|}^\infty \frac{d\lambda}{(\lambda^2 - \Delta^2 + m^2 - i\delta)^N}, \tag{3.45}$$

where we write $N = r + 1 - n/2$ for short. Another change of variables $u = \lambda^2$ leads to

$$\begin{aligned} (m^2 - \Delta^2 - i\delta)^{-N} \Big\{ \int_{\Delta^2}^\infty \frac{du}{2\sqrt{u}} \Big[\frac{u}{m^2 - \Delta^2 - i\delta} + 1 \Big]^{-N} + \\ + 2\int_0^{\Delta^2} \frac{du}{2\sqrt{u}} \Big[\frac{u}{m^2 - \Delta^2 - i\delta} + 1 \Big]^{-N} \Theta(-\Delta) \Big\}. \end{aligned} \tag{3.46}$$

These integrals can be expressed in terms of the hypergeometric functions ${}_2F_1(\alpha, \beta; \gamma; z)$ through the identities (3.39a), (3.39b) listed in section 3.3. Using the transformation formula (3.38) together with ${}_2F_1(0, \beta; \gamma; z) = {}_2F_1(\alpha, 0; \gamma; z) = 1$ we arrive at

$$\begin{aligned} I_r = \mu^\epsilon \frac{(-1)^{-r}}{\Gamma(r)} \frac{2\Gamma(N)}{(4\pi)^{\frac{n}{2}}} \Big[- \Delta(m^2 - \Delta^2 - i\delta)^{-N} {}_2F_1\Big(N, \frac{1}{2}; \frac{3}{2}; \frac{-\Delta^2}{m^2 - \Delta^2 - i\delta}\Big) \\ + \frac{\Gamma\left(N + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{(2N-1)\Gamma(N)} (m^2 - \Delta^2 - i\delta)^{\frac{1}{2}-N} \Big], \end{aligned} \tag{3.47}$$

where $N = r + 1 - n/2$.

Let us first discuss the case when r is equal to zero or negative integer, i.e., when the integrand, apart from the heavy quark propagator, is a polynomial. The integrals for the physical case $n \to 4$ are divergent, but we can make sense of it through the analytic continuation. At fixed r the integral I_r is taken to be an analytic function of the complex dimension n. For r equal to zero or negative integer and $n/2 \neq \mathbb{Z}$ all functions appearing in (3.47) are finite, apart from $\Gamma(r)$ that is infinitely large. Thus I_r vanishes for r zero or negative integer everywhere in the n complex plane apart from the points on the real axis with integer $n/2 \geqslant r+1$. Analytic continuation of I_r (3.47) is then equal to zero in the whole n plane. Integrals over polynomials (and one heavy quark propagator) are in the dimensional regularization thus equal to zero. In particular, the one-point scalar function $\bar{A}_0 = 0$.

For the two-point function we have $r = 1$ and therefore $N = \epsilon/2$. So the two point function is

$$\begin{aligned} -\frac{2}{16\pi^2} (4\pi\mu^2)^{\frac{\epsilon}{2}} \Big\{ \Gamma\Big(\frac{\epsilon}{2}\Big) (-\Delta)(m^2 - \Delta^2 - i\delta)^{-\frac{\epsilon}{2}} {}_2F_1\Big(\frac{\epsilon}{2}, \frac{1}{2}; \frac{3}{2}; \frac{-\Delta^2}{m^2 - \Delta^2 - i\delta}\Big) \\ - \pi(m^2 - \Delta^2 - i\delta)^{\frac{1}{2}} \Big\}. \end{aligned} \tag{3.48}$$

Since $\Gamma(\epsilon/2) \to 2/\epsilon - \gamma + \mathcal{O}(\epsilon)$ we have to expand hypergeometric function around $\epsilon/2 = 0$ in order to get the finite terms correctly

$$ {}_2F_1\Big(\frac{\epsilon}{2}, \frac{1}{2}; \frac{3}{2}; z\Big) = 1 + \frac{\epsilon}{2}\frac{\partial}{\partial N}\, {}_2F_1\Big(N, \frac{1}{2}; \frac{3}{2}; z\Big)\Big|_{N=0} + \dots . \tag{3.49} $$

The partial derivative can be found using the series expansion (3.36) and is

$$ \frac{\partial}{\partial N}\ {}_2F_1\Big(N, \frac{1}{2}; \frac{3}{2}; z\Big)\Big|_{N=0} = -\ln(1-z) - z^{-\frac{1}{2}}\ln\Big(\frac{1+\sqrt{z}}{1-\sqrt{z}}\Big) + 2. \tag{3.50} $$

This leads us to the final result for the two-point function

$$ \begin{aligned} &\frac{i\mu^\epsilon}{(2\pi)^{4-\epsilon}}\int d^{4-\epsilon}q \frac{1}{(v{\cdot}q - \Delta + i\delta)(q^2 - m^2 + i\delta)} = \\ &\qquad \frac{2\Delta}{(4\pi)^2}\Big\{\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln\Big(\frac{m^2}{\mu^2}\Big) + 2 - 2F\Big(\frac{m}{\Delta}\Big)\Big\}, \end{aligned} \tag{3.51} $$

where F(x) is a function as defined in [?] valid for both positive and negative Δ (while m is always taken to be positive real)

$$ F\Big(\frac{1}{x}\Big) = \begin{cases} \frac{1}{x}\sqrt{x^2-1}\ \ln(x + \sqrt{x^2-1} + i\delta)\ ; & |x| > 1, \\ -\frac{1}{x}\sqrt{1-x^2}\Big[\frac{\pi}{2} - \tan^{-1}\Big(\frac{x}{\sqrt{1-x^2}}\Big)\Big]\ ; & |x| \le 1, \end{cases} \tag{3.52} $$

with δ an infinitesimal positive parameter. Note that for $x < -1$, $F(1/x)$ has an imaginary part that corresponds to the particle creation. Also, the two point function has to be a continuous function of Δ, as can be seen from (3.41) or (3.46). It is easy to check that for $|x| = 1$ the two-point function is continuous as then $F(\pm 1) = 0$. The two-point function is also continuous for $\Delta \to 0$. Even though $F(1/x)$ diverges as $x \to 0$, the two point function (3.51) is finite and equal to $m/8\pi$.

Finally, for $r \geqslant 2$ both $\Gamma(r)$ and $\Gamma(N)$ in (3.47) are finite in the limit $n \to 4$, so that Eq. (3.47) can be used directly, with n set to $n = 4$.

3.5 Three-point scalar function

The one loop scalar three-point function with one heavy quark propagator is given by

$$ -\frac{1}{16\pi^2}\bar{C}_0(v, k, \Delta, m_1, m_2) = \frac{i\mu^\epsilon}{(2\pi)^n}\int d^n q \frac{1}{(v{\cdot}q - \Delta + i\delta)(q^2 - m_1^2 + i\delta)((q+k)^2 - m_2^2 + i\delta)}. \tag{3.53} $$

This integral is finite in 4 dimensions, so that ϵ can be set to zero. Using the Feynman parameterization (3.18) we get

$$ \bar{C}_0 = -\int_0^\infty 2d\lambda \int_0^1 dx \frac{1}{[\lambda^2 + k^2x^2 + 2v{\cdot}kx\lambda - (k^2 + m_1^2 - m_2^2)x + 2\Delta\lambda + m_1^2 - i\delta]}. \tag{3.54} $$

The integration over x can be made trivial through the change of variables $\lambda = \lambda' + \alpha x$. We choose α to be the solution of

$$ (k + \alpha v)^2 = 0, \tag{3.55} $$

as then the term quadratic in x is zero. The solution is $\alpha_{1,2} = (-v\cdot k \pm \sqrt{(v\cdot k)^2 - k^2})$ and is real for any real four-vector k^μ (this can be easily seen by going in the frame, where $v^\mu = (1,0,0,0)$ with the square root then equal to $\sqrt{\vec{k}^2}$). Changing the integration order as in (3.23), and then integrating over x, we get

$$\begin{aligned}\bar{C}_0 = -\Bigg[\int_0^\infty \frac{2d\lambda}{(A\lambda+B)} \ln\left(\frac{\lambda^2 + C\lambda + D + (A\lambda + B)}{\lambda^2 + C\lambda + D}\right) \\ + \int_0^1 \frac{2\alpha d\lambda}{(-A\alpha\lambda + B)} \Big\{ \ln[\alpha^2\lambda^2 - C\alpha\lambda + D + (-A\alpha\lambda + B)] \\ - \ln[\alpha^2\lambda^2 - C\alpha\lambda + D + (-A\alpha\lambda + B)\lambda] \Big\}\Bigg],\end{aligned} \tag{3.56}$$

where

$$A = 2(v\cdot k + \alpha), \quad B = 2\Delta\alpha + m_2^2 - m_1^2 - k^2,$$
$$C = 2\Delta, \qquad D = m_1^2 - i\delta.$$

and α one of the solutions $\alpha_{1,2}$ of the quadratic equation (3.55). In (3.56) we have used the fact that D is the only complex parameter and split the logarithm in the second integrand. The integrands of both the first and the second integral have vanishing residues. In the second integral we then add and subtract the values of the logarithms for $\lambda = B/A\alpha$ and write

$$\begin{aligned}\bar{C}_0 = -\frac{1}{A}\Bigg[\int_0^\infty \frac{2d\lambda}{\lambda + B/A} \ln\left(\frac{\lambda^2 + (A+C)\lambda + (B+D)}{\lambda^2 + C\lambda + D}\right) \\ - \int_0^1 \frac{2d\lambda}{\lambda - B/(A\alpha)} \Big(\ln\left[\alpha^2\lambda^2 - (A+C)\alpha\lambda + (B+D)\right] \\ - \ln\left[\alpha^2\lambda_0^2 - (A+C)\alpha\lambda_0 + (B+D)\right]\Big) \\ + \int_0^1 \frac{2d\lambda}{\lambda - B/(A\alpha)} \Big(\ln\left[\alpha(\alpha - A)\lambda^2 + (B - C\alpha)\lambda + D\right] \\ - \ln\left[\alpha(\alpha - A)\lambda_0^2 + (B - C\alpha)\lambda_0 + D\right]\Big)\Bigg],\end{aligned} \tag{3.57}$$

with $\lambda_0 = B/(A\alpha)$. Note, that all three integrals in (3.57) have integrands with vanishing residues. These integrals can be reduced into the sums of the dilogarithms, where care has to be taken regarding the imaginary parts of the arguments of the logarithms. The solutions of the integrals can be found in section 3.7. The solution of the first integral can be found in (3.105), (3.107), with the definitions (3.96), (3.98) (where $a_1 = a_2 = 0$, note also the minus sign), while the solutions to the last two integrals can be found in (3.89), (3.91). Using the functions S_3 and I_2 defined in section 3.7 the three-point function (3.53) finally reads

$$\begin{aligned}\bar{C}_0 = \frac{2}{A}\Big[I_2\left(0, 0, A + C, B + D, C, D, -B/A\right) \\ +S_3\left(\alpha^2, -(A + C)\alpha, B + D, B/A\alpha\right) \\ -S_3\left(\alpha(\alpha - A), B - C\alpha, D, B/A\alpha\right)\Big].\end{aligned} \tag{3.58}$$

Note that the value of the three-point function in (3.58) does not depend on which of the solutions $\alpha_{1,2}$ of the equation (3.55) is used. This can be used as a useful check in the numerical implementation.

The solution is simplified considerably if $k^2 = 0$. Then the x integration in (3.54) is trivial. Proceeding similarly as above we arrive at

$$\bar{C}_0(v,k,\Delta,m_1,m_2)\Big|_{k^2=0} = -\frac{1}{v\cdot k}\sum_i \rho(\varkappa_i)\left[\mathrm{Li}\left(\frac{\lambda_0}{\lambda_0-\varkappa_i}\right)+\frac{1}{2}\ln^2(\lambda_0-\varkappa_i)\right], \tag{3.59}$$

where $\lambda_0 = (m_1^2 - m_2^2)/(2v\cdot k)$, while $\varkappa_i$ are the solutions of

$$\begin{aligned}\lambda^2 + 2(v\cdot k+\Delta)\lambda + m_2^2 - i\delta &= (\lambda-\varkappa_1)(\lambda-\varkappa_3),\\ \lambda^2 + 2\Delta\lambda + m_1^2 - i\delta &= (\lambda-\varkappa_2)(\lambda-\varkappa_4),\end{aligned} \tag{3.60}$$

and $\rho(\varkappa_i) = (-1)^{i+1}$.

The solution is even further simplified if besides $k^2 = 0$ also $m_1 = m_2 = m$. Then

$$\bar{C}_0(v,k,\Delta,m,m)\Big|_{k^2=0} = -\frac{1}{v\cdot k}\sum_i \rho(\varkappa_i)\left[\frac{1}{2}\ln^2(-\varkappa_i)\right], \tag{3.61}$$

with $\varkappa_i$ and $\rho(\varkappa_i)$ given in (3.60). The three point function in this limit has been calculated before and is given explicitly in [?] (see Eq. (A10) of [?]). The two expressions agree completely. A number of numerical checks between numerically integrated expression (3.56) and final expression (3.58) have been performed as well.

3.6 Four-point function

The scalar four-point function with one heavy quark propagator is defined as

$$\begin{aligned}-\frac{1}{16\pi^2}\bar{D}_0(v,p_1,p_2,\Delta,m_1,m_2,m_3) =& \\ \frac{i\mu^\epsilon}{(2\pi)^n}\int\frac{d^nq}{(v\cdot q-\Delta)(q^2-m_1^2)((q+p_1)^2-m_2^2)((q+p_2)^2-m_3^2)},&\end{aligned} \tag{3.62}$$

where the $i\delta$ prescription has been omitted in the notation. Again, the integral is convergent and ϵ can be set to zero. Since the Feynman parameterization (3.18) is not symmetric in A_i and B, the elegant transformation used in the calculation of the conventional four-point function [?] and further improved in [?] unfortunately cannot be applied. Instead, one repeatedly uses the propagator identity (3.19) to solve the integral (3.62). Since the parameter α in the propagator identity (3.19) has to be real, the calculation differs depending on the values of the external momenta p_1 and p_2.

First we take up the case, when one of the following inequalities is true $p_{1,2}^2 \geqslant (m_1+m_{2,3})^2$ or $p_{1,2}^2 \leqslant (m_1-m_{2,3})^2$. If necessary, we renumber the momenta and reshuffle the propagators in (3.62) in such a way that either $p_1^2 \geqslant (m_1+m_2)^2$ or $p_1^2 \leqslant (m_1-m_2)^2$ in order to simplify the discussion. Then we use the propagator identity (3.19) on the second and the third propagators of (3.62)

$$\begin{aligned}\frac{1}{(q^2-m_1^2+i\delta)((q+p_1)^2-m_2^2+i\delta)} =& \frac{1-\alpha}{[(q+p_1)^2-m_2^2+i\delta][(q+l)^2-M^2+i\delta]}\\ &+\frac{\alpha}{[q^2-m_1^2+i\delta][(q+l)^2-M^2+i\delta]},\end{aligned} \tag{3.63}$$

where α is an arbitrary parameter and

$$l = \alpha p_1, \tag{3.64}$$
$$M^2 = (1-\alpha)m_1^2 + \alpha m_2^2 - \alpha(1-\alpha)p_1^2. \tag{3.65}$$

We choose α such that $M^2 = 0$. This is satisfied by real α if either $p_1^2 \geqslant (m_1+m_2)^2$ or $p_1^2 \leqslant (m_1-m_2)^2$ as has been assumed above. The scalar four point function is then

$$\begin{aligned}&\frac{i}{(2\pi)^4}\int d^4q \frac{1-\alpha}{(v{\cdot}q-\Delta)[(q+p_1)^2-m_2^2][(q+p_2)^2-m_3^2][(q+l)^2+i\delta]} + \\ &\frac{i}{(2\pi)^4}\int d^4q \frac{\alpha}{(v{\cdot}q-\Delta)[q^2-m_1^2][(q+p_2)^2-m_3^2][(q+l)^2+i\delta]},\end{aligned} \tag{3.66}$$

with α being the solution of

$$p_1^2\alpha^2 + (m_2^2 - m_1^2 - p_1^2)\alpha + m_1^2 = 0. \tag{3.67}$$

To calculate the two integrals in (3.66) it suffices to consider

$$\begin{aligned}&-\frac{1}{16\pi^2}\tilde{D}_0(v,k_1,k_2,k_3,\Delta,M_1,M_2) = \\ &\qquad \frac{i}{(2\pi)^4}\int d^4q \frac{1}{(v{\cdot}q-\Delta)[(q+k_1)^2-M_1^2][(q+k_2)^2-M_2^2][(q+k_3)^2+i\delta]},\end{aligned} \tag{3.68}$$

where the $i\delta$ prescription has not been written out explicitly in the first three propagators.

Using the Feynman parameterization (3.18) and integrating over q we arrive at

$$\begin{aligned}\tilde{D}_0 = \int_0^\infty 2d\lambda \int_0^1 dx \int_0^x dy[&p_{23}x^2 + p_{12}y^2 + (p_{13}-p_{23}-p_{12})xy + \lambda^2 + (P_2-P_3)x\lambda + P_3\lambda \\ &+ (P_1-P_2)y\lambda + (-p_{23}+M_2^2)x + (p_{23}-p_{13}+M_1^2-M_2^2)y - i\delta]^{-2},\end{aligned} \tag{3.69}$$

with

$$\begin{aligned}p_{ij} &= (k_i-k_j)^2, \\ P_i &= 2(v{\cdot}k_i+\Delta).\end{aligned} \tag{3.70}$$

To simplify the integration we introduce new variables $y = xy'$ and $\lambda = x\lambda'$. The integration limits are then $\int_0^1 dx \int_0^x dy \int_0^\infty 2d\lambda \to \int_0^1 dx \int_0^1 xdy' \int_0^\infty x2d\lambda'$. Since x is positive and δ an infinitesimal parameter of which only the sign matters, the extra factor of x^2 in the numerator can be canceled against the similar factor in the denominator. After the cancellation the denominator is linear in x. The integration over x is now trivial and yields

$$\begin{aligned}\tilde{D}_0 = \int_0^\infty 2d\lambda \int_0^1 dy&[(p_{23}-p_{13}+M_1^2-M_2^2)y + P_3\lambda - p_{23} + M_2^2 - i\delta]^{-1} \times \\ &[p_{12}y^2 + \lambda^2 + (P_1-P_2)y\lambda + P_2\lambda + (-p_{12}+M_1^2-M_2^2)y + M_2^2 - i\delta]^{-1}.\end{aligned} \tag{3.71}$$

To cancel the y^2 term in the integral above a new variable $\lambda = \lambda' + \beta y$ is introduced, with β chosen to solve

$$\beta^2 + (P_1-P_2)\beta + p_{12} = 0. \tag{3.72}$$

The solutions are real, if $(k_2-k_1)^\mu$ is real (cf. Eq. (3.70)). Since both k_1 and k_2 are taken to be real four-vectors, this is always the case. We then get

$$\tilde{D}_0 = \Big(\int_0^\infty 2d\lambda \int_0^1 dy + \int_{-\beta}^0 2d\lambda \int_{-\lambda/\beta}^1 dy \Big) \frac{1}{[a_1 y + b_1\lambda + c_1][a_2 y + b_2 \lambda + c_2 + d_2 y\lambda + \lambda^2]}, \tag{3.73}$$

with

$$\begin{aligned} a_1 &= \beta P_3 + p_{23} - p_{13} + M_1^2 - M_2^2, & a_2 &= \beta P_2 + M_1^2 - M_2^2 - p_{12},\\ b_1 &= P_3, & b_2 &= P_2,\\ c_1 &= M_2^2 - p_{23} - i\delta, & c_2 &= M_2^2 - i\delta,\\ & & d_2 &= 2\beta + P_1 - P_2. \end{aligned} \tag{3.74}$$

After y integration we arrive at

$$\begin{aligned} \tilde{D}_0 =& \frac{1}{(\lambda_1-\lambda_2)} \frac{1}{(b_1 d_2 - a_1)} \Bigg[\\ & \int_0^\infty \frac{2d\lambda}{\lambda-\lambda_1} \Big\{ \ln\Big(\frac{b_1\lambda + c_1}{b_1\lambda + a_1 + c_1}\Big) - \ln\Big(\frac{\lambda^2 + b_2\lambda + c_2}{\lambda^2 + (b_2+d_2)\lambda + a_2 + c_2}\Big) \Big\} \\ -& \int_0^\infty \frac{2d\lambda}{\lambda-\lambda_2} \Big\{ \ln\Big(\frac{b_1\lambda + c_1}{b_1\lambda + a_1 + c_1}\Big) - \ln\Big(\frac{\lambda^2 + b_2\lambda + c_2}{\lambda^2 + (b_2+d_2)\lambda + a_2 + c_2}\Big) \Big\} \\ -& \int_0^1 \frac{2d\lambda}{\lambda+\lambda_1/\beta} \Big\{ \ln\Big(\frac{(a_1-\beta b_1)\lambda + c_1}{-\beta b_1 \lambda + a_1 + c_1}\Big) - \ln\Big(\frac{\beta(\beta-d_2)\lambda^2 + (a_2 - b_2\beta)\lambda + c_2}{\beta^2\lambda^2 - \beta(b_2+d_2)\lambda + a_2 + c_2}\Big) \Big\} \\ +& \int_0^1 \frac{2d\lambda}{\lambda+\lambda_2/\beta} \Big\{ \ln\Big(\frac{(a_1-\beta b_1)\lambda + c_1}{-\beta b_1 \lambda + a_1 + c_1}\Big) - \ln\Big(\frac{\beta(\beta-d_2)\lambda^2 + (a_2 - b_2\beta)\lambda + c_2}{\beta^2\lambda^2 - \beta(b_2+d_2)\lambda + a_2 + c_2}\Big) \Big\} \Bigg], \end{aligned} \tag{3.75}$$

with $\lambda_{1,2}$ the solutions of

$$(b_1 d_2 - a_1)\lambda^2 + (a_2 b_1 - a_1 b_2 + c_1 d_2)\lambda + a_2 c_1 - a_1 c_2 = (b_1 d_2 - a_1)(\lambda-\lambda_1)(\lambda-\lambda_2). \tag{3.76}$$

Note that the integrands above have vanishing residues, i.e., arguments of the two logarithms in the integrands are the same for $\lambda = \lambda_{1,2}$. Note as well, that the infinitesimal imaginary parts of c_1 and c_2, the $-i\delta$ in (3.74), have to be equal. They originate from the same infinitesimal parameter in (3.69) that after the integration over x appears twice in (3.71). The size of the infinitesimal parts of $\lambda_{1,2}$ compared to $-i\delta$ are thus unambiguously defined and have to be kept track of until the end of the calculation.

The integrals (3.75) can be expressed in terms of the dilogarithms. This has been done in section 3.7. The solution for the first two integrals can be found in (3.105), with the definitions in (3.96), (3.98), while the solution for the last two integrals can be found in (3.94), (3.95). Together with Eqs. (3.66), (3.68) and the cascade of abbreviations (3.74), (3.72), (3.70) this gives the complete solution of the four point function with at least one external momentum p_1, p_2 satisfying $p_{1,2}^2 \geqslant (m_1+m_{2,3})^2$ or $p_{1,2}^2 \leqslant (m_1-m_{2,3})^2$. Collecting the terms and rearranging the last two propagators in (3.62) if necessary, the four point function for $p_1^2 \geqslant (m_1+m_2)^2$ or $p_1^2 \leqslant (m_1-m_2)^2$ finally reads

$$\begin{aligned} \bar{D}_0(v, p_1, p_2, \Delta, m_1, m_2, m_3) =& (1-\alpha)\tilde{D}_0(v, p_1, p_2, l, \Delta, m_2, m_3) \\ &+ \alpha\, \tilde{D}_0(v, 0, p_2, l, \Delta, m_1, m_3), \end{aligned} \tag{3.77}$$

with $l = \alpha p_1$ and α the solution of $p_1^2\alpha^2 + (m_2^2 - m_1^2 - p_1^2)\alpha + m_1^2 = 0$, while

$$\begin{aligned}
\tilde{D}_0(v,k_1,k_2,k_3,\Delta,M_1,M_2) &= \frac{1}{(\lambda_1-\lambda_2)}\frac{2}{(b_1d_2-a_1)}\Big[\\
& I_2\left(-c_1/b_1,-(a_1+c_1)/b_1,b_2,c_2,b_2+d_2,a_2+c_2,\lambda_1\right) \\
& -I_2(-c_1/b_1,-(a_1+c_1)/b_1,b_2,c_2,b_2+d_2,a_2+c_2,\lambda_2) \\
& -I_1(\beta^2-\beta d_2,\beta^2,a_2-\beta b_2,-\beta(b_2+d_2),a_1-\beta b_1,-\beta b_1,c_2,a_2+c_2,c_1,a_1+c_1,-\lambda_1/\beta) \\
& +I_1(\beta^2-\beta d_2,\beta^2,a_2-\beta b_2,-\beta(b_2+d_2),a_1-\beta b_1,-\beta b_1,c_2,a_2+c_2,c_1,a_1+c_1,-\lambda_2/\beta)\Big],
\end{aligned} \tag{3.78}$$

with $a_1 \ldots d_2$ defined in (3.74), β defined in (3.72), p_{ij}, P_i defined in (3.70) and $\lambda_{1,2}$ solutions of (3.76). Note that parameters α and β are solutions of quadratic equations (3.67) and (3.72) that in general have two real solutions each. The value of the four point function $\bar{D}_0(v,k_1,k_2,k_3,\Delta,M_1,M_2)$ does not depend on which of the solutions are chosen in the evaluation of (3.77), (3.78). This fact can be used as a useful test in the numerical implementation of the expressions given above.

Now we take up the special case of p_1 on the light-cone, i.e., $p_1^2 = 0$. From (3.67) it follows that $\alpha = m_1^2/(m_1^2 - m_2^2)$. If $m_1^2 \neq m_2^2$ then α is finite and the calculation proceeds as before, (3.62)-(3.75). For equal masses m_1 and m_2, we evaluate the integrals by first taking $m_1^2 \neq m_2^2$ and then performing the limit $m_1^2 \to m_2^2$ and thus $|\alpha| \to \infty$. The external momenta in the last propagators of (3.66) are both equal to $l = \alpha p_1$. Thus the last external momentum in $\tilde{D}_0$ of (3.68) is going to be $k_3 = l = \alpha p_1$ for both of the integrals in (3.66). In the limit $\alpha \to \infty$ the following leading order values (3.74) are obtained: $a_1 \to 2\alpha p_1\cdot(\beta v - p_2)$, $b_1 \to 2\alpha v\cdot p_1$, $c_1 \to 2\alpha p_1\cdot p_2$, where we have used also the fact that $p_1^2 = 0$. The other coefficients a_2, b_2, c_2, d_2 do not depend on α. The first term in the denominator of the integrand in (3.73) is then proportional to α, while the second term in the denominator does not depend on α at all. The α in the denominator cancels against the α in the numerator of (3.66). For the case of equal masses $m_1^2 = m_2^2$ and $p_1^2 = 0$, the solution is then the same as for the case $m_1^2 \neq m_2^2$ except that (i) one has to replace $(1-\alpha)$ and α in (3.66) with -1 and 1 respectively, and that (ii) a_1, b_1, c_1 in (3.74), (3.75) have to be replaced by their limiting values (divided by α)

$$\begin{aligned}
a_1 &\to a_1^l = 2p_1\cdot(\beta v - p_2), \\
b_1 &\to b_1^l = 2v\cdot p_1, \\
c_1 &\to c_1^l = 2p_1\cdot p_2 - i\delta,
\end{aligned} \tag{3.79}$$

where β is the solution to (3.72) (with P_i defined in (3.70)), and is different for the two integrals in (3.66). In the limiting value of c_1 coefficient given in (3.79) an additional $-i\delta$ prescription has been added. As will be shown in the next paragraph, this does not have any effect on the value of the four-point function. It does make possible, however, to express the integrals in (3.75) in terms of the functions I_1 and I_2 as in (3.78).

It is easy to see, that the limiting procedure as explained above does lead to an unambiguous result. One might in principle worry that limits $m_1^2 \to m_2^2$ taken from above and below, corresponding to the limits $\alpha \to \infty$ and $\alpha \to -\infty$ respectively, would lead to different results. The question is most conveniently settled if the $\tilde{D}_0$ functions in Eq. (3.77) are replaced by the expressions given in Eq. (3.71). Once the limit $m_1 \to m_2$ is taken, the first factors of the integrands have the same limiting value. For α large, thus the leading term is

$$\begin{aligned}\bar{D}_0 \to -\alpha\int_0^\infty 2d\lambda\int_0^1 & dy[-2\alpha p_1{\cdot}p_2 y + 2\alpha v{\cdot}p_1\lambda + 2\alpha p_1{\cdot}p_2 - i\delta]^{-1}\times \\ & [(p_1-p_2)^2y^2+\lambda^2+2v\cdot(p_1-p_2)y\lambda + \\ & \quad + P_2\lambda + (-(p_1-p_2)^2+m_2^2-m_3^2)y+m_3^2-i\delta]^{-1} \\ +\alpha\int_0^\infty 2d\lambda\int_0^1 & dy[-2\alpha p_1{\cdot}p_2 y + 2\alpha v{\cdot}p_1\lambda + 2\alpha p_1{\cdot}p_2 - i\delta]^{-1}\times \\ & [p_2^2y^2+\lambda^2-2v\cdot p_2 y\lambda + \\ & \quad + P_2\lambda + (-p_2^2+m_1^2-m_3^2)y+m_3^2-i\delta]^{-1},\end{aligned} \tag{3.80}$$

with $P_2 = 2(v{\cdot}p_2+\Delta)$. After collecting the two integrands in (3.80) the first factor in the integrands cancels and one finds

$$\begin{aligned}\bar{D}_0 \to -\int_0^\infty 2d\lambda\int_0^1 & ydy[(p_1-p_2)^2y^2+\lambda^2+2v\cdot(p_1-p_2)y\lambda + \\ & \quad + P_2\lambda + (-(p_1-p_2)^2+m_2^2-m_3^2)y+m_3^2-i\delta]^{-1}\times \\ & [p_2^2y^2+\lambda^2-2v\cdot p_2 y\lambda + \\ & \quad + P_2\lambda + (-p_2^2+m_1^2-m_3^2)y+m_3^2-i\delta]^{-1},\end{aligned} \tag{3.81}$$

This result exhibits clearly the fact that (i) the limit $m_1^2 \to m_2^2$ is independent of whether it is taken from above or below and (ii) the limit is independent of the size (or even the sign) of the infinitesimal parameter in the first terms of the integrands in (3.80).

When the momenta p_1, p_2 satisfy $(m_1-m_{2,3})^2 < p_{1,2}^2 < (m_1+m_{2,3})^2$, rendering a complex α, the procedure outlined above in (3.62)-(3.77) cannot be applied directly. Starting from (3.62), we then use the propagator identity (3.19) on the last two propagators in (3.62), where we set α such that $l^2 = (p_1+\alpha(p_2-p_1))^2 = 0$. This has a real solution for α since p_1 and p_2 are timelike as has been assumed at the beginning of this paragraph. Changing the notation slightly we then have for the scalar four point function (omitting the $-i\delta$ prescription in the notation)

$$\begin{aligned}&\frac{i}{(2\pi)^4}\int d^4q\frac{\alpha'}{(v{\cdot}q-\Delta)[q^2-m_1^2][(q+p_1)^2-m_2^2][(q+l)^2-M^2]} + \\ &\frac{i}{(2\pi)^4}\int d^4q\frac{1-\alpha'}{(v{\cdot}q-\Delta)[q^2-m_1^2][(q+p_2)^2-m_3^2][(q+l)^2-M^2]},\end{aligned} \tag{3.82}$$

with α' the (real) solution of

$$(p_1+\alpha'(p_2-p_1))^2 = 0, \tag{3.83}$$

and

$$l = p_1+\alpha'(p_2-p_1), \tag{3.84a}$$
$$M^2 = (1-\alpha')m_2^2+\alpha' m_3^2-\alpha'(1-\alpha')(p_2-p_1)^2. \tag{3.84b}$$

The integrals in (3.82) can now be solved using the procedure outlined above (3.62)-(3.79), once we permute the last two propagators with l taking the role of p_1 in (3.62). Note also, that α' solves quadratic equation (3.83) that in general has two solutions. The final result for the four point function $\bar{D}_0$ does not depend on which of the two solutions is taken in (3.82). This fact can be exploited in the numerical implementation as a useful check.

The four point function has already been calculated before for a special case of $m_1 = m_2 = m_3$, $p_1^2 = p_2^2 = 0$ and $p_1^\mu - p_2^\mu = Mv^\mu$ (see Eq. (A11) of [?]). It has been checked numerically that the two solutions, the one given here and the solution of [?], agree for this special case. A number of other numerical tests have been performed. The direct numerical integration of (3.69) and the evaluation of analytical result given above have been found to agree numerically. It has been also checked that the results do not depend on which of the two solutions for α, β or α' is taken. The solution for the four-point function calculated above has also been checked numerically to have the branch cuts as required by analyticity and unitarity.

The five-point as well as the higher-point functions can be expressed in terms of the scalar functions given above using the standard procedure [?, ?]. Consider for instance the case of five-point scalar function. This is a function of four vectors, v and p_1, p_2, p_3. The five-point function is first multiplied by $v_\mu \epsilon_{\alpha\beta\gamma\delta}$ and then antisymmetrized in all five indices. The resulting tensor is zero, because there is no antisymmetrical tensor with five indices in four dimensions. Then the tensor is multiplied first with $p_{1\alpha} p_{2\beta} p_{3\gamma} q_\delta$ and finally with $v^\mu \epsilon_{\alpha'\beta'\gamma'\delta'} p_1^{\alpha'} p_2^{\beta'} p_3^{\gamma'} q^{\delta'}$. Using the decomposition of the product of two Levi-Civita tensors in terms of the Kronecker delta functions and expressing the scalar products $q{\cdot}p_i$ in terms of the propagators $((q+p_i)^2 - m_{i+1}^2)$ and $(q^2 - m_1^2)$, the five-point function can be expressed in terms of the four point functions. The tensor functions can also be expressed in terms of the scalar functions using the algebraic reduction [?].

In the numerical implementation of the expressions as given here, further care has to be taken regarding the numerical instabilities. Such numerical instabilities can for instance arise, if one of the solutions of the quadratic equation is much smaller than its coefficients. There is also a possibility of a cancellation between the dilogarithmic functions, when the values of the dilogarithms separately are much larger then their sum. These difficulties can be dealt with along the lines of Ref. [?].

3.7 Reduction to dilogarithms

In this section we will express the integrals appearing in (3.57), (3.75) in terms of the dilogarithms. First we review the derivations given in [?]. Consider

$$\begin{aligned} R(\lambda_1, \lambda_0) &= \int_0^1 d\lambda \frac{1}{\lambda - \lambda_0} [\ln(\lambda - \lambda_1) - \ln(\lambda_0 - \lambda_1)] \\ &= \int_{-\lambda_1}^{1-\lambda_1} d\lambda \frac{1}{\lambda - \lambda_0 + \lambda_1} [\ln \lambda - \ln(\lambda_0 - \lambda_1)], \end{aligned} \tag{3.85}$$

where $\lambda_{0,1}$ may be complex. The residue of the pole of the integrand is zero. The cut of the logarithm is along the negative real axis, so for λ_1 not real, the cut is outside the triangle 0, $-\lambda_1$, $1 - \lambda_1$. The integration path can thus be deformed to (for λ_1 real, this statement is trivial)

$$\int_{-\lambda_1}^{1-\lambda_1} d\lambda = \int_0^{1-\lambda_1} d\lambda - \int_0^{-\lambda_1} d\lambda.$$

Making the substitutions $\lambda = (1 - \lambda_1)\lambda'$ and $\lambda = \lambda_1 \lambda'$ we obtain

$$\begin{aligned} R(\lambda_1, \lambda_0) = &\int_0^1 d\lambda \Big[\frac{d}{d\lambda} \ln\Big(1 + \lambda \frac{1-\lambda_1}{\lambda_1 - \lambda_0}\Big)\Big] [\ln \lambda(1 - \lambda_1) - \ln(\lambda_0 - \lambda_1)] \\ &- \int_0^1 d\lambda \Big[\frac{d}{d\lambda} \ln\Big(1 - \lambda \frac{\lambda_1}{\lambda_1 - \lambda_0}\Big)\Big] [\ln(-\lambda\lambda_1) - \ln(\lambda_0 - \lambda_1)]. \end{aligned} \tag{3.86}$$

Since λ is positive real, none of the arguments of the logarithms crosses the negative real axis. After integration per parts

$$\begin{aligned} R(\lambda_1,\lambda_0) = \mathrm{Li}\left(\frac{\lambda_1-1}{\lambda_1-\lambda_0}\right) &+ \ln\left(\frac{1-\lambda_0}{\lambda_1-\lambda_0}\right)[\ln(1-\lambda_1)-\ln(\lambda_0-\lambda_1)] \\ -\mathrm{Li}\left(\frac{\lambda_1}{\lambda_1-\lambda_0}\right) &- \ln\left(\frac{-\lambda_0}{\lambda_1-\lambda_0}\right)[\ln(-\lambda_1)-\ln(\lambda_0-\lambda_1)]\,. \end{aligned} \tag{3.87}$$

This can be further simplified using (3.34b)

$$\begin{aligned} R(\lambda_1,\lambda_0) =& \mathrm{Li}\left(\frac{\lambda_0}{\lambda_0-\lambda_1}\right) + \left[\eta\left(-\lambda_1,\frac{1}{\lambda_0-\lambda_1}\right) + 2\pi i\Re^{(-)}(\lambda_0-\lambda_1)\right]\ln\frac{\lambda_0}{\lambda_0-\lambda_1} \\ &- \mathrm{Li}\left(\frac{\lambda_0-1}{\lambda_0-\lambda_1}\right) - \left[\eta\left(1-\lambda_1,\frac{1}{\lambda_0-\lambda_1}\right) + 2\pi i\Re^{(-)}(\lambda_0-\lambda_1)\right]\ln\frac{\lambda_0-1}{\lambda_0-\lambda_1}, \end{aligned} \tag{3.88}$$

with η defined in (3.27) and $\Re^{(-)}(x)$ defined in (3.25). Note that this result differs slightly from the one in [?] as it is defined also for the arguments lying on the negative real axis. The extension to negative real arguments was not necessary in [?] as then the λ_0 was always real. This is not the case in the calculation of the four point function with one heavy quark propagator, as the λ_1 and λ_2 in (3.75) can have nonzero imaginary parts. The momenta and the masses in the calculation can then be chosen such, that one of the arguments appearing in (3.88) can lie on the negative real axis.

Next we turn to the integral

$$S_3(a,b,c,\lambda_0) = \int_0^1 d\lambda \frac{1}{\lambda-\lambda_0}[\ln(a\lambda^2+b\lambda+c)-\ln(a\lambda_0^2+b\lambda_0+c)], \tag{3.89}$$

with a real, while b, c, λ_0 may be complex but such, that the imaginary part of the argument of the logarithm does not change sign for $x \in [0,1]$ (also $\Im(c) \neq 0$).

Let ϵ and δ be infinitesimal quantities that have the *opposite* sign from the imaginary part of first and second argument of the logarithm respectively. That is, the signs of the arguments are as given by $-i\epsilon$ and $-i\delta$. Then

$$\begin{aligned} S_3 =& \int_0^1 d\lambda \frac{1}{\lambda-\lambda_0}[\ln(\lambda-\lambda_1)(\lambda-\lambda_2)-\ln(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)] \\ &- \eta\left(a-i\epsilon,\frac{1}{a-i\delta}\right)\ln\left(\frac{\lambda_0-1}{\lambda_0}\right), \end{aligned} \tag{3.90}$$

with $\lambda_{1,2}$ the solutions of $a\lambda^2+b\lambda+c = a(\lambda-\lambda_1)(\lambda-\lambda_2)$. Next we split up the logarithms, use the fact that the imaginary part of $(\lambda-\lambda_1)(\lambda-\lambda_2)$ has the same sign as the imaginary part of c/a and use the definitions of $R(\lambda_1,\lambda_0)$ (3.85) to get

$$\begin{aligned} S_3(a,b,c,\lambda_0) =& R(\lambda_1,\lambda_0) + R(\lambda_2,\lambda_0) \\ &+ \left[\eta(-\lambda_1,-\lambda_2) - \eta(\lambda_0-\lambda_1,\lambda_0-\lambda_2) - \eta\left(a-i\epsilon,\frac{1}{a-i\delta}\right)\right]\ln\frac{\lambda_0-1}{\lambda_0}, \end{aligned} \tag{3.91}$$

with ϵ and δ defined before Eq. (3.90).

For future reference we also define

$$S_2(b,c,\lambda_0) = \int_0^1 d\lambda \frac{1}{\lambda-\lambda_0}[\ln(b\lambda+c)-\ln(b\lambda_0+c)], \tag{3.92}$$

with b real and c, λ_0 possibly complex ($\Im(c) \neq 0$). Defining as above infinitesimal parameters ϵ' and δ' to have signs *opposite* to the imaginary parts of the first and the second argument of the logarithms respectively, we obtain

$$S_2(b,c,\lambda_0) = R(-c/b,\lambda_0) - \eta\Big(b - i\epsilon', \frac{1}{b-i\delta'}\Big) \ln\frac{\lambda_0 - 1}{\lambda_0}. \tag{3.93}$$

Next we turn to the integrals appearing in the calculations of the three-point and four-point functions with one heavy quark propagator. Consider first

$$\begin{aligned} &I_1(a_1,a_2,b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4,\lambda_0) = \\ &\qquad \int_0^1 d\lambda \frac{1}{\lambda-\lambda_0}\Big[\ln\frac{b_3\lambda+c_3}{b_4\lambda+c_4} - \ln\frac{a_1\lambda^2+b_1\lambda+c_1}{a_2\lambda^2+b_2\lambda+c_2}\Big], \end{aligned} \tag{3.94}$$

with $a_{1,2}$ and $b_{1,\cdots,4}$ real, while λ_0, $c_{1\cdots4}$ may be complex but such that $\Im(c_1)\Im(c_2) > 0$ and $\Im(c_3)\Im(c_4) > 0$. Also, the coefficients are such, that for $\lambda = \lambda_0$ the two logarithms are equal, so that the residue of the integrand is equal to zero. Such an integral appears in the calculation of the four-point scalar function (3.75). To reduce the integral I_1 to the integrals S_2, S_3 we add and subtract the values of the logarithms at the pole. Since the numerators and the denominators of the logarithms in (3.94) have imaginary parts of the same sign, we can split the logarithms. Additional η terms appear, however, when we split the logarithms with λ set to λ_0. As the result we get

$$\begin{aligned} I_1 = {} & S_2(b_3,c_3,\lambda_0) - S_2(b_4,c_4,\lambda_0) - S_3(a_1,b_1,c_1,\lambda_0) + S_3(a_2,b_2,c_2,\lambda_0) \\ & + \Big[\eta\Big(a_1\lambda_0^2+b_1\lambda_0+c_1, \frac{1}{a_2\lambda_0^2+b_2\lambda_0+c_2}\Big) + 2\pi i \Re^{(-)}(a_2\lambda_0^2+b_2\lambda_0+c_2) \\ & \quad - \eta\Big(b_3\lambda_0+c_3, \frac{1}{b_4\lambda_0+c_4}\Big) - 2\pi i\Re^{(-)}(b_4\lambda_0+c_4)\Big] \ln\frac{\lambda_0-1}{\lambda_0}. \end{aligned} \tag{3.95}$$

with η defined in (3.27) and $\Re^{(-)}$ in (3.25).

Next consider the integral

$$\begin{aligned} &I_2(a_1,a_2,g_1,f_1,g_2,f_2,\lambda_0) = \\ &\qquad \int_0^\infty d\lambda \frac{1}{\lambda-\lambda_0}\left\{\ln\frac{\lambda-a_1}{\lambda-a_2} - \ln\frac{\lambda^2+g_1\lambda+f_1}{\lambda^2+g_2\lambda+f_2}\right\}, \end{aligned} \tag{3.96}$$

with $g_{1,2}$ real, while λ_0, $a_{1,2}$, $f_{1,2}$ may be complex with the restriction $\Im(a_1)\Im(a_2) > 0$, $\Im(f_1)\Im(f_2) > 0$. Then the logarithms can be split without introducing η terms, independent of the value of λ as long as this is real. Also the arguments of the logarithms in (3.96) are taken to be the same for $\lambda = \lambda_0$, so that the residue of the integrand is zero. Such integrals appear in the calculation of the three-point function (3.57) and in the calculation of the four-point function (3.75). We rewrite the integral (3.96) as

$$I_2 = \int_0^\infty d\lambda \frac{1}{\lambda-\lambda_0}\Big\{\ln\frac{\lambda-a_1}{\lambda-a_2} - \ln\frac{(\lambda-b_1)(\lambda-b_2)}{(\lambda-c_1)(\lambda-c_2)}\Big\}, \tag{3.97}$$

with

$$\begin{aligned} \lambda^2+g_1\lambda+f_1 &= (\lambda-b_1)(\lambda-b_2), \\ \lambda^2+g_2\lambda+f_2 &= (\lambda-c_1)(\lambda-c_2), \end{aligned} \tag{3.98}$$

where $\Im(b_1)\Im(b_2) < 0$, $\Im(c_1)\Im(c_2) < 0$, $\Im(b_1 b_2)\Im(c_1 c_2) > 0$ as can be seen from the constraints on $g_{1,2}$, $f_{1,2}$. Then the logarithms can be split up in the sum of the logarithms with arguments linear in λ.

To the integral (3.97) we add logarithms with λ set to λ_0 and then split the logarithms

$$\begin{aligned} 0 &= \ln \frac{\lambda_0 - a_1}{\lambda_0 - a_2} - \ln \frac{(\lambda_0 - b_1)(\lambda_0 - b_2)}{(\lambda_0 - c_1)(\lambda_0 - c_2)} \\ &= \sum_i \rho(\varkappa_i) \ln(\lambda_0 - \varkappa_i) - \eta', \end{aligned} \tag{3.99}$$

where $\varkappa_i$ are the coefficients $a_{1,2}$, $b_{1,2}$, $c_{1,2}$ with $\rho(\varkappa_i) = 1$ for $a_1, c_{1,2}$ and $\rho(\varkappa_i) = -1$ for $a_2, b_{1,2}$. There is also a sum of η terms that we do not write out explicitly, but just denote by η', as it will be reabsorbed in the final result. Note also, that in the case of $\lambda_0 - \varkappa_i$ real and negative the logarithm is calculated using the prescription $\lambda_0 - \varkappa_i \to \lambda_0 - \varkappa_i + i\delta$, with δ a positive infinitesimal parameter (see also (3.24)-(3.28)). The integral is then

$$I_2 = \sum_i \rho(\varkappa_i) \int_0^\infty d\lambda \frac{1}{\lambda - \lambda_0} [\ln(\lambda - \varkappa_i) - \ln(\lambda_0 - \varkappa_i)] + \eta' \int_0^\infty \frac{d\lambda}{\lambda - \lambda_0}. \tag{3.100}$$

The separate integrals are divergent so they have to be regulated. We use the cutoff M that is sent to infinity at the end of the calculation. Note also, that there is no problem with the pole in the last term even if λ_0 is real, as then η' is zero.

The regulated integrals are then

$$\int_0^M d\lambda \frac{1}{\lambda - \lambda_0} [\ln(\lambda - \varkappa_i) - \ln(\lambda_0 - \varkappa_i)]. \tag{3.101}$$

Let us from here on first assume, that $\lambda_0 - \varkappa_i$ is not negative real. Changing the variable $\lambda = M\lambda'$ and using the calculation of $R(\lambda_1, \lambda_0)$ (3.85), (3.88) we get

$$\begin{aligned} &\int_0^1 d\lambda \frac{1}{\lambda - \frac{\lambda_0}{M}} \left[\ln\left(\lambda - \frac{\varkappa_i}{M}\right) - \ln\left(\frac{\lambda_0}{M} - \frac{\varkappa_i}{M}\right) \right] = \\ &\quad \mathrm{Li}\,\frac{\lambda_0}{\lambda_0 - \varkappa_i} - \mathrm{Li}\,\frac{\lambda_0 - M}{\lambda_0 - \varkappa_i} + \eta\Big(-\varkappa_i, \frac{1}{\lambda_0 - \varkappa_i}\Big) \ln \frac{\lambda_0}{\lambda_0 - \varkappa_i} - \eta\Big(1 - \frac{\varkappa_i}{M}, \frac{M}{\lambda_0 - \varkappa_i}\Big) \ln \frac{\lambda_0 - M}{\lambda_0 - \varkappa_i}. \end{aligned} \tag{3.102}$$

For M big enough the last term is zero. The M dependent dilogarithm can be transformed using relation (3.34a)

$$\mathrm{Li}\,\frac{-M}{\lambda_0 - \varkappa_i} = -\,\mathrm{Li}\,\frac{\lambda_0 - \varkappa_i}{-M} - \frac{1}{6}\pi^2 - \frac{1}{2}\ln^2\left(\frac{M}{\lambda_0 - \varkappa_i}\right). \tag{3.103}$$

The argument of the dilogarithm on the right-hand side goes toward zero as $M \to \infty$, so that in that limit the dilogarithm vanishes. Next we split the logarithm in the last term and write

$$\ln^2\left(\frac{M}{\lambda_0 - \varkappa_i}\right) = \ln^2 M - 2\ln M \ln(\lambda_0 - \varkappa_i) + \ln^2(\lambda_0 - \varkappa_i). \tag{3.104}$$

The first term gives zero once summed over in (3.100), while the second term cancels against the η' term in (3.100). Leaving the case of $\lambda_0 - \varkappa_i$ negative real to the reader, the final result is

$$\begin{aligned} I_2 = \sum_i \rho(\varkappa_i) \Big\{ &\mathrm{Li}\,\frac{\lambda_0}{\lambda_0 - \varkappa_i} + \left[\eta\Big(-\varkappa_i, \frac{1}{\lambda_0 - \varkappa_i}\Big) + 2\pi i \Re^{(-)}(\lambda_0 - \varkappa_i) \right] \ln \frac{\lambda_0}{\lambda_0 - \varkappa_i} \\ &+ \frac{1}{2}\ln^2(\lambda_0 - \varkappa_i) - \ln(\lambda_0 - \varkappa_i)\ln(-\lambda_0) \Big\}, \end{aligned} \tag{3.105}$$

with

$$\rho(\varkappa_i) = \begin{cases} +1; \varkappa_i = a_1, c_{1,2}, \\ -1; \varkappa_i = a_2, b_{1,2}. \end{cases} \tag{3.106}$$

Note that this solution applies also for the case encountered in the calculation of the three point function (3.57), when $a_1 = a_2 = 0$. Then the terms containing $a_{1,2}$ cancel each other, so they can be dropped altogether for the case of Eq. (3.57).

There is one more point worth mentioning regarding the expression (3.105). One might think that problems could arise for $\lambda_0 - \varkappa_i$ negative real or $\lambda_0/(\lambda_0 - \varkappa_i)$ real as then one has to deal with the cuts in the logarithm and the dilogarithmic function[§]. We use the prescription for the arguments lying exactly on the cuts of the functions as described before Eq. (3.24) and after Eq. (3.33). One could as well use a different prescription, with infinitesimal parameter ϵ in (3.24), (3.33) taken to be negative, and with appropriately adjusted η and $\Re^{(-)}$ functions. It has been checked numerically, that the result (3.105) does not change, if the alternative prescription is used. Thus the result (3.105) is valid for any complex λ_0, $\varkappa_i$ independent of the prescription used for the arguments lying on the cut.

For the special case of λ_0 real the result (3.105) simplifies considerably. The η term is then zero. Also the last term in (3.105), that arises from the η' term in (3.100), then sums up to zero. For λ_0 real we have

$$I_2 = \sum_i \rho(\varkappa_i) \Big[\mathrm{Li} \frac{\lambda_0}{\lambda_0 - \varkappa_i} + \frac{1}{2} \ln^2(\lambda_0 - \varkappa_i) \Big], \tag{3.107}$$

with $\varkappa_i$ and $\rho(\varkappa_i)$ as in (3.106).

Of special interest is the case of λ_0 almost real, i.e., $\lambda_0 = \lambda_0^{re} + i\delta'$, where λ_0^{re} is the real part of λ_0 and δ' an infinitesimal (not necessarily positive) parameter. One can of course still use the solution (3.105). The problem is, however, that for both $\varkappa_i$ and λ_0 almost real one has to keep track of the relative sizes of the infinitesimal imaginary parts. This complication can be avoided by the following procedure. First we set λ_0 in the second line of (3.99) equal to its real part. By doing this, the arguments of the logarithms can cross the negative real axis, which is compensated by a new sum of η functions, η'. Then instead of (3.100) we have

$$\sum_i \rho(\varkappa_i) \int_0^\infty d\lambda \frac{1}{\lambda - \lambda_0^{re} - i\delta'} [\ln(\lambda - \varkappa_i) - \ln(\lambda_0^{re} - \varkappa_i)] + \eta' \int_0^\infty \frac{d\lambda}{\lambda - \lambda_0}. \tag{3.108}$$

In the first integral δ' can be safely put equal to zero as the resulting integrand has vanishing residue, with the logarithm in the numerator being an analytic function in some neighbourhood of the pole (since $\Im(\varkappa_i) \neq 0$). The integral thus does not depend on how we avoid the pole (i.e. δ' can be positive, negative or zero). In the second integral one has to keep the imaginary part of λ_0.

The final result for almost real $\lambda_0 = \lambda_0^{re} + i\delta'$ is then

$$I_2 = \sum_i \rho(\varkappa_i) \Big[\mathrm{Li} \frac{\lambda_0^{re}}{\lambda_0^{re} - \varkappa_i} + \frac{1}{2} \ln^2(\lambda_0^{re} - \varkappa_i) - \ln(\lambda_0^{re} - \varkappa_i) \ln(-\lambda_0) \Big], \tag{3.109}$$

where in the last logarithm λ_0 is kept together with its infinitesimal imaginary part.

[§]Note that there exists such a combination of parameters v, $p_{1,2}$, Δ and $m_{1,2,3}$ in (3.62) that $\lambda_0 - \varkappa_i$ in (3.105) is negative real for some i, as can be seen from definition of $a_1, \ldots, d_2$ (3.74), definition of $\lambda_{1,2}$ (3.76) and the expression for the four-point function (3.78).

Chapter 4

Weak interactions in the effective theory approach

In this chapter we will briefly review the standard methods used in the phenomenology of weak interactions, the operator product expansion (OPE) and the renormalization group (RG) equations. These are used to arrive at a set of local operators describing weak interactions at low energies. At the end the factorization approximation, that is used to evaluate hadronic matrix elements of the current-current local operators, is described.

4.1 Operator Product Expansion

In this section we will briefly review the ideas behind the Operator Product Expansion (OPE) and its application to weak interactions. The original idea dates back to Wilson [**?**], who conjectured that the divergent part of a product of two operators could be described by a sum of local operators $Q_n(x)$

Let us note on passing that the perturbative proof of OPE has been given by Zimmerman [**?**], while a nonperturbative proof can be found in [**?**]. The operator product expansion in general reads

The application of the OPE to weak interactions comes from the observation that the distances at which weak interactions occur are set by the mass of the intermediate W and Z bosons, i.e., $x - y \sim 1/m_W$. If one is interested in the processes at energy scales μ much smaller than the weak scale, $\mu \ll m_W$, or, in other words, in the processes, that effectively occur at typical distances $1/\mu$ that are much larger than $x - y \sim 1/m_W$, we can take the limit $x \to y$ (or equivalently $m_W \to \infty$) and use the operator product expansion.

Let us formulate this in some more detail. We start from the charged current part of the weak Lagrangian

The scattering matrix S_{fi} to the first nonzero order in perturbation theory for a process involving four quarks is

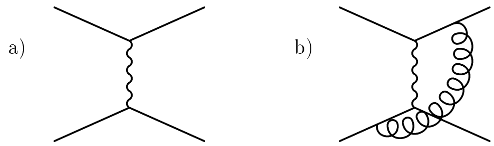


Figure 4.1: The tree level diagram in the full theory a), that is translated to the Q_2 operator in the effective theory. The operator Q_1 arises from the QCD interactions b). In addition to the diagram b) also the diagrams with gluons attached to different legs appear.

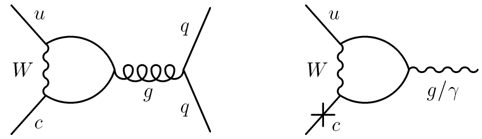


Figure 4.2: The diagrams that give rise to the penguin operators. The diagram on the left gives rise to the operators $Q_{3....,6}$, while the operator to the right gives rise to the magnetic penguin operators $Q_{7,8}$. Cross denotes mass insertion.

Actually, the effective Lagrangian (**??**) is valid only in the absence of QCD interactions. Once these are taken into account, as shown on Fig. 4.1, another four-quark operator appears in the OPE, $Q_1 = (\bar{u}^\alpha d^\beta)_{V-A}(\bar{s}^\beta c^\alpha)_{V-A}$, where summation over color indices α, β is understood. To arrive at this operator, the following identity for the $SU(N_c)$ generators is used

If one is interested in the processes involving a slightly different change of flavors, other operators can appear as well. For instance, if the s-quark in (**??**), (**??**) is replaced by a d quark (or vice versa), an additional set of diagrams, shown on Fig. 4.2, is possible at $\mathcal{O}(\alpha_s)$ order and gives rise to the "penguin" operators $Q_{3,...6}$. The complete set of operators for $\Delta C = 1$, $\Delta S = 0$ (neglecting electroweak penguins) is then

$$Q_1^d = (\bar{u}^\alpha d^\beta)_{V-A}(\bar{d}^\beta c^\alpha)_{V-A}, \qquad Q_2^d = (\bar{u}d)_{V-A}(\bar{d}c)_{V-A}, \tag{4.1a}$$

$$Q_1^s = (\bar{u}^\alpha s^\beta)_{V-A}(\bar{s}^\beta c^\alpha)_{V-A}, \qquad Q_2^s = (\bar{u}s)_{V-A}(\bar{s}c)_{V-A}, \tag{4.1b}$$

$$Q_3 = (\bar{u}c)_{V-A}\sum_q (\bar{q}q)_{V-A}, \qquad Q_4 = (\bar{u}^\alpha c^\beta)_{V-A}\sum_q (\bar{q}^\beta q^\alpha)_{V-A}, \tag{4.1c}$$

$$Q_5 = (\bar{u}c)_{V-A}\sum_q (\bar{q}q)_{V+A}, \qquad Q_6 = (\bar{u}^\alpha c^\beta)_{V-A}\sum_q (\bar{q}^\beta q^\alpha)_{V+A}, \tag{4.1d}$$

where we have suppressed the color indices in the currents of the form $(\bar{q}q') = (\bar{q}^\alpha q'^\alpha)$, while the sum over q runs over the active quark-flavors. At the scale $\mu \simeq m_c$ these are $q = u, d, s, c$. In the following chapters, we will be interested also in the final states involving photons and lepton pairs. For these decays the set of the relevant operators is enlarged by the magnetic penguins (corresponding to the diagrams on Figure 4.2)

$$Q_7 = \frac{e}{4\pi^2} m_c F_{\mu\nu} \bar{u}\sigma^{\mu\nu} P_R c, \qquad Q_8 = \frac{g_s}{4\pi^2} m_c G^a_{\mu\nu} \bar{u}\sigma^{\mu\nu} T^a P_R c, \tag{4.2}$$

with g_s the strong coupling constant, and the semileptonic operators (corresponding to the diagrams on Fig. 4.3)

$$Q_9 = \frac{e^2}{16\pi^2}(\bar{u}_L \gamma^\mu c_L)(\bar{l}\gamma_\mu l), \qquad Q_{10} = \frac{e^2}{16\pi^2}(\bar{u}_L \gamma^\mu c_L)(\bar{l}\gamma_\mu \gamma_5 l), \tag{4.3}$$

where $q_L = P_L q$ and $P_{L,R} = \frac{1}{2}(1 \pm \gamma_5)$ are the chirality projection operators.

4.2 Renormalization Group and OPE

In the calculation of the Wilson coefficients typically expressions of the form $\alpha_s \ln(\mu/m_W)$ appear. Here μ is a typical scale at which the processes occur. For processes involving the decay of c-quark a typical scale is of order 1 GeV. The ratio of scales in the argument of the logarithm is thus very large, of order 100, and consequentially the factor $\alpha_s \ln\left(\mu/m_W\right)$ is of order $\mathcal{O}(1)$. Even though the QCD coupling α_s is not terribly large at the scales of around 1 GeV and could be used as a perturbative expansion parameter, the appearance of large logarithms prevents the straightforward application of perturbation theory. All large logarithms of the form $\left(\alpha_s \ln\left(\mu/m_W\right)\right)^n$ have to be summed up using renormalization group equations, if one wants to get the correct leading order expression for the Wilson coefficients at lower energy scales.

The RG evolution is done in several steps [**?**]. First the Wilson coefficients C_i are calculated at the weak scale $\mu \sim m_W$ to some given order in the perturbative QCD expansion. For instance, at the leading order $C_2(m_W) = 1$, while $C_1 = C_{3,\ldots,6} = 0$ (the coefficients $C_{7,9,10}$ will be discussed later on). Then to the same order anomalous dimensions of the four-quark operators in the effective theory are calculated (at the leading order this is to the order α_s). These are then used to evolve the Wilson coefficients to lower energy scales.

Let us first introduce the notion of the anomalous dimensions*. To do so, consider first an invariant amplitude A for a given process. Assume that the calculation of the invariant amplitude A in the full theory is known. This has to be the same to the value of A obtained in the effective theory, i.e., by using the operator product expanded effective Lagrangian. Using LSZ theorem the amplitude for the four-quark scattering is proportional to $Z_q^2 \langle Q_i^{(0)} \rangle_0$, where Z_q is the renormalization constant for the quark field $q^{(0)} = Z_q^{1/2} q$, while $\langle Q_i^{(0)} \rangle_0$ is the amputated Green function of the unrenormalized operator. However, $Z_q^2 \langle Q_i^{(0)} \rangle_0$ is still divergent, so that additional multiplicative operator renormalization has to be introduced

It is illuminating to consider also a different point of view, closer to the conventional renormalization in terms of the coupling constants. Instead of absorbing the divergences in the renormalizations of operators, these can be absorbed in the "coupling constants", the Wilson coefficients C_i. The renormalized Wilson coefficients are thus

The evolution of the Wilson coefficient is now easily determined. Following the usual notation [**?**], first the anomalous dimensions matrix γ is introduced

*We will follow closely the introduction given in [**?**].

To get some flavor for the effects of the RGE, we show the leading order RG evolution of a single operator

In general, the Wilson coefficients at lower scale are calculated through the following steps [?]. First the Wilson coefficients $C_i(m_W)$ at weak scale are calculated by matching the effective theory with five active flavors $q = u, d, s, c, b$ onto the full theory. Then the anomalous dimensions $\gamma^{(5)}$ are calculated in the effective theory with five flavors. Using $\gamma^{(5)}$, Wilson coefficients are evolved down to the scale of b-quark, obtaining $C_i(m_b)$. If one is interested in the processes at lower scales, e.g at the charm quark scale, b-quark is integrated out as an effective degree of freedom. This is accomplished by matching the effective theory with five flavors onto the effective theory with four flavors. The remaining Wilson coefficients are then evolved down to the charm scale using the anomalous dimension matrices of the four-flavor effective theory. Thus

Let us be more specific and discuss the case of $\Delta C = 1$, $\Delta S = 0$ charm decays in some more detail. The effective Lagrangian at the weak scale $\mu \sim m_W$ is

Regarding the RG evolution of the operators $Q_{1,..,10}$ there are several important things to note. First of all Q_{10} does not mix with other operators due to chirality. Furthermore, it has vanishing anomalous dimension, so that $C_{10}(\mu_c) = C_{10}(m_W)$. Next, the dimension five operators $Q_{7,8}$ do not mix into the dimension six operators $Q_{1,...,6}$ and Q_9. If one is interested in these operators solely, the dimension five operators can be dropped from the RG analysis. We will follow this procedure and evaluate C_7 separately. Note also, that (i) Q_9 operator does not mix into the operators $Q_{1,...,6}$ and (ii) the penguin operators $Q_{3,...,6}$ do not mix into the operators $Q_{1,2}$. One can thus consider the RG evolution of the reduced operator basis $Q_{1,2}$, $Q_{1,...,6}$ or $Q_{1,...,9}$, if one is interested in smaller sets of the Wilson coefficients $C_{1,2}$, $C_{1,...,6}$, or $C_{1,...,9}$, without introducing any error in the calculation. Finally, it is convenient to introduce a rescaled operator $\tilde{Q}_9 = \alpha/\alpha_s(\bar{u}c)_{V-A}(\bar{l}l)_V$, as then the anomalous dimension depends only on the strong coupling, and can be expanded as in (??). The calculation of the Wilson coefficients then proceed as outlined above.

It is instructive to do the α_s counting. At the leading order the RG evolution sums terms of the form $\alpha_s \ln(m_c^2/m_W^2)$, which are numerically of the order $\mathcal{O}(1)$. At the leading order one thus has to start with the initial values $C_i(m_W)$ calculated at α_s^0, and then evolve them using the 1 loop anomalous dimensions (i.e. of order α_s) to get the order $\mathcal{O}(1)$ values $C_i(\mu)$ at lower scales. Going to higher orders, an additional power of α_s is added at each step. We thus have

We start a more quantitative discussion with the values of the Wilson coefficients to the order $\mathcal{O}(\alpha_s)$ at the weak scale. These are known for quite some time and are in the naive dimensional regularization scheme (NDR)[†] [?]

$$C_1(m_W) = \frac{11}{2}\frac{\alpha_s(m_W)}{4\pi}, \qquad C_2(m_W) = 1 - \frac{11}{6}\frac{\alpha_s(m_W)}{4\pi}, \tag{4.4}$$

[†]In the naive dimensional regularization the Dirac matrices are assumed to obey $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, where $g_{\mu\nu}$ is a $4-\epsilon$ dimensional metric tensor. The γ_5 matrix is assumed to commute with the Dirac matrices $\{\gamma_\mu, \gamma_5\} = 0$ [?].

while $C_{3,\ldots,9}(m_W) = 0$. Since above μ_b the penguin operators do not enter the effective Lagrangian due to the unitarity of the CKM matrix, the Wilson coefficients $C_{1,2}$ in (**??**) can be evolved down to $\mu \sim \mu_b$ using the 2×2 anomalous dimension matrix (which can be found in [**?**] or in Eq. (5.12) of [**?**]). At the scale μ_b the b-quark is integrated out, i.e., the five-flavor effective theory (**??**) is matched onto the four-flavor theory given by

The sets of operators $\{Q^d_{1,2}, Q_{3,\ldots,6}, Q_9\}$ and $\{Q^s_{1,2}, Q_{3,\ldots,6}, Q_9\}$ from the first line of (**??**) are then evolved to the charm scale $\mu \sim m_c$ using the 7×7 anomalous dimension matrices $\gamma^{(4)}$ for the four quark effective theory. The 6×6 LO and NLO submatrices involving the gluonic penguins are listed in Eqs. (6.25), (6.26) of Ref. [**?**] and have been calculated in [**?**, **?**]. The remaining entries are listed in Eqs (8.11), (8.12) of Ref. [**?**] and have been calculated in [**?**].

In summary, the RG evolution from $\mu_W \sim m_W$ to μ_c for the $\Delta C = 1$ transitions is described by the following procedure

$$m_b < \mu < m_W : \qquad \vec{C}(\mu) = U_5(\mu, m_W)\vec{C}(m_W), \tag{4.5}$$

$$\mu = m_b : \qquad \vec{C}(m_b) \to \vec{Z}(m_b), \tag{4.6}$$

$$m_c < \mu < m_b : \qquad \vec{C}(\mu) = U_4(\mu, m_b)\vec{Z}(m_b), \tag{4.7}$$

with U_5 and U_4 the 2×2 and 7×7 evolution matrices for five and four active flavors respectively. They can be found in Eqs. (3.93)-(3.98) of [**?**]. The $Z(m_b)$ are given in (**??**)-(**??**). The values of the Wilson coefficients are listed in Table 4.1. For a comparison the values of the Wilson coefficients at the leading order are given as well, but calculated with the two-loop evolution of the strong coupling constant (**??**). The values are given for the central value of $\Lambda^{(5)} = 216 \pm 25$ MeV and $m_b = 4.25$ GeV. The one sigma change in $\Lambda^{(5)}$ corresponds to a change of about 10% in $C_{1,\ldots,6}$. We find a pronounced scale dependence for the C_9 coefficient below 1.5 GeV, as a consequence of the large cancelations in the RG evolution equations. The situation is very similar to the case of the coefficient Z_{7V} in $K_L \to \pi^0 e^+ e^-$ [**?**]. The LO value of C_9 even changes sign near $\mu \sim 1$ GeV, being positive for $\mu > 1$ GeV.

-	μ(GeV)	C_1	C_2	C_3	C_4	C_5	C_6	$\tilde{C}_9$	C_9
LO	1.0	-0.64	1.34	0.016	-0.036	0.010	-0.046	-0.0013	-0.07
NLO	1.0	-0.49	1.26	0.024	-0.060	0.015	-0.060	-0.011	-0.60
NLO	1.5	-0.37	1.18	0.013	-0.036	0.012	-0.033	-0.0018	-0.13
NLO	2.0	-0.30	1.14	0.009	-0.025	0.009	-0.021	-0.0016	-0.13

Table 4.1: Values of Wilson coefficients at scales $\mu = 1, 1.5, 2$ GeV, calculated at the next-to-leading order (NLO) as explained in the text. For a comparison in the first line the LO values are given at the scale $\mu = 1$ GeV, but calculated with the two loop evolution of the strong coupling constant (**??**). In the last column the properly scaled $C_9 = 8\pi/\alpha_s(\mu)\tilde{C}_9$ Wilson coefficient is given.

Note, however, that the penguin operators are proportional to the $V^*_{cb}V_{ub}$ matrix elements (**??**). In the Wolfenstein parametrization (**??**) this is $\sim \lambda^5$, which has to be compared to the CKM suppression of the $Q_{1,2}$ operators, $V^*_{cs}V_{us} \sim \lambda$, where $\lambda = \sin\theta_c = 0.22$. Penguin operators are thus suppressed by $\lambda^4 \sim 10^{-3}$ in the $\Delta C = 1$ transitions, even more so because the penguin Wilson coefficients are of the order $C_{3,\ldots,6}(m_c) \leq 10^{-1} C_{1,2}(m_c)$ as shown in Table 4.1 (see also [**?**, **?**]).

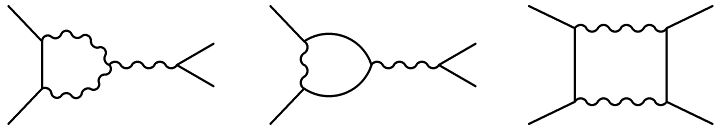


Figure 4.3: The penguin and box diagrams contributing to $c \to u l^+ l^-$ decay at the quark level.

The penguin operators in the $\Delta C = 1$ transitions are not relevant numerically, except in special observables such as CP asymmetries [?]. They are thus neglected in the following.

Note, that also the Q_7, Q_9 operators are suppressed by a factor $\lambda^4 \sim 10^{-3}$ compared to the $Q_{1,2}$ operators. They will, however, be kept in the analysis, because of possibly large non-SM contributions that will be discussed in more detail in chapter ??. In the SM they are, however, negligible. Incidentally this also means, that the uncertainties in the value of the C_9 coefficient, observed above, will not propagate into the decay rates.

It is interesting to compare the $\Delta C = 1$ transition discussed above with the $\Delta B = 1$, $b \to s$ transition. The relevant effective Lagrangian at the μ_b scale is [?]

Let us now conclude the discussion of the $\Delta C = 1$ transitions by turning to the magnetic penguin operator Q_7 and the semileptonic operator Q_{10}. The value of the C_7 Wilson coefficient is obtained following the same procedure as outlined in Eqs. (4.5)-(4.7), using the operator basis $Q_{1,\dots,7}$. The major difference compared to the case of RG evolution with the Q_9 operator is, that the leading order mixing of the operators $Q_{7,8}$ with the operators $Q_{1,\dots,6}$ vanishes. It is only at the two-loop level, that the anomalous dimension matrix has nonzero values mixing $C_{1,\dots,6}$ into C_7. The expansion of C_7 in powers of α_s then begins at the order $\mathcal{O}(1)$, contrary to the case of C_9 (??). The factor e in Q_7 also insures that the expansion of $\gamma(\alpha_s)$ in (??) is unchanged, with the difference, that $\gamma_{i7}^{(0)}$ receive contributions from the two-loop calculation. Since the two-loop results are scheme dependent, so is $\gamma^{(0)}$. It is then customary to introduce the effective anomalous dimension matrix $\gamma^{(0)\mathrm{eff}}$ [?], which is scheme independent, as is the case for the leading order results. Using the LO anomalous dimension matrix $\gamma^{(0)\mathrm{eff}}$, the NLO evolution for α_s, $m_b = 4.25$ GeV, the result is (see also [?])

It is instructive to compare the values of $C_{7,9}$ Wilson coefficients obtained from the RG analysis with the invariant amplitudes that one would get from the full electroweak theory, but by neglecting the QCD interactions (i.e. by evaluating the diagrams of Fig. 4.3). The invariant amplitudes of the QCD neglected calculation have the same structure as is obtained from the effective Lagrangian (??) when used *at tree level*. The parameters of the invariant amplitudes obtained by neglecting the QCD contributions will be denoted by $C_{7,9}^{\mathrm{IL}}$ (with IL standing for Inami, Lim [?]). It is important to stress that these are *not the Wilson coefficients*, as they only parametrize the invariant amplitudes. However, based on the (unproved) expectations, that C_9 is not much changed by the QCD corrections, C_9^{IL} has been often used in the literature as an estimate for $C_9(\mu)$ [?, ?].

The values of the parameters $C_{7,9,10}^{\mathrm{IL}}$ are easily obtained from the calculation of Ref. [?] for the $b \to s l^+ l^-$ transitions. Following [?] we find, that the coefficients are of the form

The important thing to note is, that the variable $x_j = m_{q_j}^2/m_W^2$ is very small for $q_j = d, s, b$. The functions $\bar{C}$ and $\bar{\Gamma}_z$ are proportional to $\bar{C}, \bar{\Gamma}_z \propto x_j$ and are thus very small. The function $\bar{F}_1$, on the other hand, is to the leading order $\bar{F}_1(x_j, x_d) \sim \frac{2}{3} Q_q \ln(x_j/x_d)$ which is of the order $\mathcal{O}(1)$. We thus arrive at

Similarly one could determine the value of C_7 at the weak scale, arriving at the leading order expression

Finally, the leading order expression in terms of $x_j = m_{q_j}^2/m_W^2$ for the C_{10}^{IL} coefficient (**??**), is

For the sake of completeness we write down at the end the effective Lagrangian of the weak interactions induced at the scales $\mu \sim m_c$, containing the operators relevant for the processes, that will be considered in chapters 5, **??**

4.3 Factorization approximation

As discussed in the previous section, the weak interactions can be described at low energies by means of an effective Lagrangian obtained through the operator product expansion and the renormalization group evolution. The effective Lagrangian for the Cabibbo allowed transitions is

The factorization approximation is a very simple but extremely useful and quite successful approximation [**?**]. In this approach the currents appearing in the operators $Q_{1,2}$ are assumed to factor. Each of the currents is proportional to interpolating stable or quasistable hadronic fields. The approximation comes in, when these interpolating full hadronic fields are approximated in one or both of the currents by an asymptotically free hadronic field, i.e., by the "in" and "out" fields. The effective interaction (**??**) is then

$where$

$(\bar{q}q)_{V-A}^H$ are the hadronized $V-A$ currents. For instance

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \big[a_1 (\bar{u}d)_{V-A}^H (\bar{s}c)_{V-A}^H + a_2 (\bar{u}c)_{V-A}^H (\bar{s}d)_{V-A}^H \big].$$

where $(\bar{q}q)_{V-A}^H$ are the hadronized $V-A$ currents. For instance

$withthedotsrepresentingotherhadronicfieldswiththesamequantumnumbers.Intheapproachadoptedhere, thehad$

a$_{1,2}$ in (4.3) are in principle unknown coefficients that have to be estimated from the experimental data. For instance, for the case of the D meson two-body nonleptonic decays $D \to P_1 P_2$, the decay amplitudes in the (naive) factorization approximation read

$(\bar{u}d)^H_{V-A} = -f\partial_\mu \pi^- + \cdots$

with the dots representing other hadronic fields with the same quantum numbers. In the approach adopted here, the hadronized current containing charm quark is obtained using the heavy quark symmetry and is given in (2.46) plus terms coming from fields with the same quantum numbers. The *effective* Wilson coefficients $a_{1,2}$ in (4.3) are in principle unknown coefficients that have to be estimated from the experimental data. For instance, for the case of the D meson two-body nonleptonic decays $D \to P_1 P_2$, the decay amplitudes in the (naive) factorization approximation read

$$\begin{aligned} M^W_{P_1P_2,D} = \\ &\frac{G_F}{\sqrt{2}} V^*_{cs} V_{ud} \Big[a_1 \Big(\langle P_2|(\bar{u}d)_\mu|0\rangle\langle P_1|(\bar{s}c)^\mu|D\rangle + \langle P_1|(\bar{u}d)_\mu|0\rangle\langle P_2|(\bar{s}c)^\mu|D\rangle + \\ &\langle P_1P_2|(\bar{u}d)_\mu|0\rangle\langle 0|(\bar{s}c)^\mu|D\rangle \Big) + a_2 \Big(\langle P_2|(\bar{s}d)_\mu|0\rangle\langle P_1|(\bar{u}c)^\mu|D\rangle + \\ &\langle P_1|(\bar{s}d)_\mu|0\rangle\langle P_2|(\bar{u}c)^\mu|D\rangle + \langle P_1P_2|(\bar{s}d)_\mu|0\rangle\langle 0|(\bar{u}c)^\mu|D\rangle \Big) \Big]. \end{aligned} \tag{4.10}$$

This is then compared with the experimental data on the Cabibbo allowed decays $D \to K\pi$, arriving at the values $a_1 \approx 1.3 \pm 0.1$, $a_2 \approx -0.55 \pm 0.1$ [?, ?]. In this analysis the final state interactions have to be taken into account. The naive factorization approximation as explained above, is taken to be valid only in the weak vertex, for the so called *bare* amplitudes. The outgoing hadronic states then interact strongly, which can lead to elastic and inelastic rescattering effects. Because D mesons lie close to the resonance region, a number of s and t channel resonances can in principle contribute. These effects are especially important for the $K\pi$, $K\eta^{(\prime)}$ states because of the presence of $S = 1$ scalar meson resonance $K^0(1950)$ (for more details see [?, ?, ?, ?, ?, ?, ?, ?, ?, ?], for the final state interactions in the $D \to PV$ decays see also [?, ?, ?] and for the doubly Cabibbo suppressed $D \to K\pi$ decays [?]).

On the other hand, starting from the weak Lagrangian (??) one can expect that

The idea of factorization has recently received a lot of attention due to the theoretical work of two groups, the approach of the QCD factorization [?, ?, ?] and the pQCD approach [?, ?, ?, ?, ?]. The underlying physical picture of these approaches is the idea of color transparency [?, ?, ?], which is effective in the heavy meson decays with energetic final decay products. As an example consider the case of $B \to \pi\pi$. The fast moving final mesons produced by the point-like source (the local operators in the OPE expansion) decouple from the soft QCD interactions. Contributions of the soft gluons are suppressed by $\Lambda_{\rm QCD}/m_b$. The QCD factorization approach gives rigorous results valid in the heavy quark limit to the leading power in $\Lambda_{\rm QCD}/m_b$, but to all orders in the perturbation theory. These ideas have been further developed for the heavy-to-heavy transitions in [?]. The application of the above formalism is, however, not possible for the D meson nonleptonic decays as here the energy release is much smaller than in the case of B mesons. Nevertheless, the factorization procedure has been applied to the nonleptonic D decays in a phenomenologically successful way as discussed above. In this sense nonleptonic D decays are halfway between B and K nonleptonic decays. Namely, it is well known that the factorization does not work in the nonleptonic K decays [?, ?, ?, ?].

Chapter 5

Nonfactorizable contributions to the decay mode $D^0 \to K^0\bar{K}^0$

The decay mechanism of the weak nonleptonic D^0 decays has motivated numerous studies, e.g., [?, ?, ?, ?, ?, ?, ?, ?]. For the nonleptonic decays of D mesons, as well as for K's and B's, the *factorization* hypothesis explained in section 4.3 has commonly been used. In this section we discuss nonfactorizable contributions to D decays, in particular in the decay mode $D^0 \to K^0\bar{K}^0$. This decay mode has been advertised as an interesting probe of the nonperturbative physics in weak decays long time ago [?]. Additional motivation to consider this decay mode comes from the recent experimental searches for the CP violating asymmetry in $D^0 \to K_S K_S$ [?].

In D decays the factorization hypothesis works reasonably well, if one is interested in an order of magnitude estimate, but it does not reproduce experimental data completely. For example, a naive application of the factorization in the charm decays leads to the rates for the $D^0 \to \pi^0\bar{K}^0$, $D^0 \to \pi^0\pi^0$, $D^0 \to K^+K^-$, $D^0 \to \pi^+\pi^-$ decays which are too strongly suppressed (see, e.g., [?, ?, ?, ?]). Consideration of either the final state interactions through resonant or nonresonant rescattering and/or of other nonfactorizable mechanisms is thus mandatory. Moreover, and this is the important point of the present chapter, in $D^0 \to K^0\bar{K}^0$ a naive application of factorization misses completely, predicting a vanishing branching ratio, in contrast with the experimental situation.

To see this, note that at tree level the $D^0 \to K^0\bar{K}^0$ decay might occur through two annihilation diagrams [?] with either $c \to s$ or $c \to d$ transition. However, they cancel each other by the GIM mechanism. Moreover, in the factorization limit, the amplitude is proportional to

$$\langle K^0\bar{K}^0|V_\mu|0\rangle\langle 0|A^\mu|D^0\rangle \simeq (p_{K^0} - p_{\bar{K}^0})_\mu\, f_D p_D^\mu = 0. \tag{5.1}$$

In many of the studies (e.g. [?, ?, ?, ?, ?]) this decay has been understood as a result of the final state interactions (FSI). In the analysis of Ref. [?] the rescattering mechanism included K^+K^- and $\pi^+\pi^-$ states leading to a branching ratio $\mathrm{Br}(D^0 \to K^0\bar{K}^0) = \frac{1}{2}\mathrm{Br}(D^0 \to K^+K^-)$. Experimental data on the other hand are [?] $\mathrm{Br}(D^0 \to K^0\bar{K}^0) = (7.1 \pm 1.9) \times 10^{-4}$ and $\mathrm{Br}(D^0 \to K^+K^-) = (4.12 \pm 0.14) \times 10^{-3}$. A recent investigation of the $D^0 \to K^0\bar{K}^0$ decay mode performed in [?] has focused on the s channel and t channel one particle exchange contributions. The s channel contribution has been taken into account through a poorly known scalar meson $f_0(1710)$ and was found to be very small, while the one particle t-exchanges yielded higher contributions, with pion exchange being the highest. In the approach of [?] the $D^0 \to K^0\bar{K}^0$ decay was realized through the scalar glueball or glue-rich scalar meson.

We will adopt the approach of the effective Lagrangians as explained in chapter 2. The subsequent analysis has been published in [?, ?]. Because the $\mathcal{O}(p)$ (factorizable) contribution is zero

(5.1), we will try to approach to the $D^0 \to K^0\bar{K}^0$ decay systematically to $\mathcal{O}(p^3)$. We do this by including first the nonfactorizable contributions coming from the chiral loops. In the weak vertex the factorization hypothesis will be used, leading to the weak transitions of the type $D^0 \to \pi^+\pi^-$ and $D^0 \to K^+K^-$ (see Figs. 5.1-5.4). In this sense the approach is similar to the factorization hypothesis as put forward in Ref. [?] for the Cabibbo allowed D decays. In Ref. [?] factorization was assumed for the weak vertex, leading to the *bare* amplitudes, that are then modified by the FSI. The final state interactions correspond to the diagram F_4 on Figure 5.4. A number of additional chiral loop diagrams will be considered in this approach. In addition, we consider the gluon condensate contributions, also of $\mathcal{O}(p^3)$, which we calculate within the Heavy-Light Chiral Quark Model (HLχQM) framework. The HLχQM is an extension of the effective Lagrangian approach of chapter 2, as it models also the interactions of light pseudoscalars and heavy mesons with quarks. By integrating out the quark degrees of freedom one is able to reproduce the effective Lagrangians of chapter 2. A brief introduction to the HLχQM will be given in section 5.2, while a more detailed description can be found in [?].

We emphasize that one cannot a priori expect for the chiral expansion to work to a good precision in the process $D \to K\bar{K}$, because the energy release is $p = 788$ MeV and hence p/Λ_χ (for $\Lambda_\chi \geq 1$ GeV) is close to unity. However, the leading contributions, that we will consider, do turn out to describe the data reasonably well. The next to leading $\mathcal{O}(p^5)$ terms might be almost of the same order of magnitude compared to the $\mathcal{O}(p^3)$ terms, with a weak suppression of the order p^2/Λ_χ^2. On the other hand, the inclusion of $\mathcal{O}(p^5)$ order in this framework is not straightforward. Before doing loop calculations at that order, one has to find a reliable framework to include light resonances ρ, K^*, $a_0(980)$, $f_0(975)$, etc. Usually the light resonances are treated using hidden gauge symmetry (see, e.g., [?]). This is not easily reconciled with the chiral perturbation theory. Even if the light resonances were included in the effective Lagrangian, one would face the problem of determining their couplings to the rest of the heavy and light states. The poorly known scalar resonances would introduce a rather large uncertainty [?]. Right now, the consistent calculation of this or higher orders does not seem to be possible. Still, the amplitude of the $D^0 \to K^0\bar{K}^0$ decay, calculated within our framework to the order $\mathcal{O}(p^3)$ turns out to be in agreement with the experimental result. Note also, that $1/m_Q$ terms have been omitted in the calculation.

5.1 Chiral loop contributions

As discussed above, in the factorization limit there are no contributions to the $D^0 \to K^0\bar{K}^0$ decay at tree level (5.1). The observation of a partial decay width $\mathrm{Br}(D \to K^0\bar{K}^0) = (7.1 \pm 1.9) \times 10^{-4}$ on the other hand implies, that we can expect sizable contributions at the one loop level. Calculations to one loop in the framework of combined chiral perturbation theory and heavy quark symmetry, the Heavy Hadron Chiral Perturbation Theory (HHχPT), involve a construction of the most general effective Lagrangian, that has the correct symmetry properties, in order to make the renormalization work. This construction together with the chiral counting has been explained in chapter 2.

The weak Lagrangian relevant for the $D^0 \to K^0\bar{K}^0$ decay is

The loop diagrams are divergent and have to be regulated. We work both in the strict $\overline{\mathrm{MS}}$ renormalization scheme, where we put $\bar{\Delta} = \frac{2}{\epsilon} - \gamma + \ln(4\pi) + 1 \to 1$ in the loop calculations as well as in the Gasser-Leutwyler (GL) renormalization scheme $\bar{\Delta} \to 0$. The first choice is the same as the one made by Stewart in [?], while the other was made by the authors of Ref. [?]. The

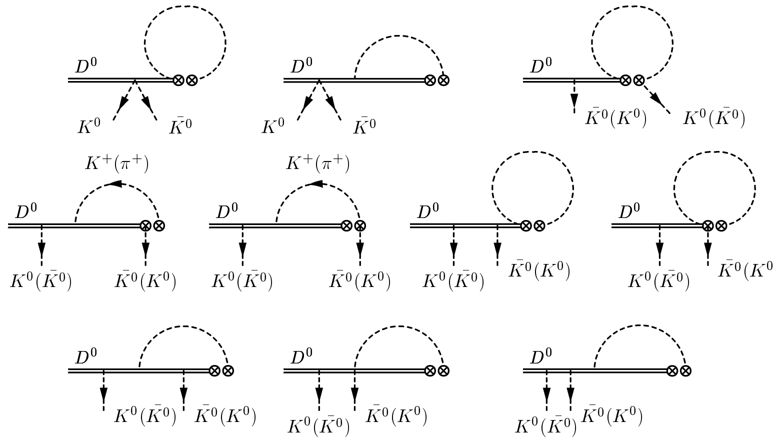


Figure 5.1: Diagrams, that give zero contribution, since the relevant vertices appearing in the heavy meson chiral Lagrangian (2.43) are zero. The double line represents heavy meson D or D^*, while dashed lines denote pseudo-Goldstone bosons.

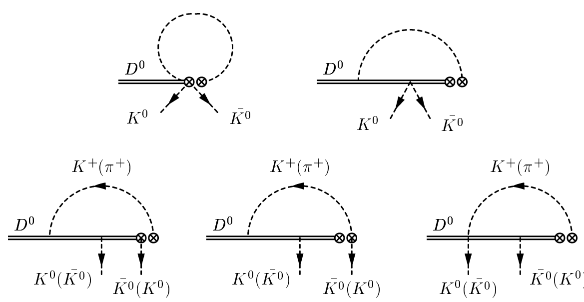


Figure 5.2: Diagrams, that give zero contributions, since the loop integrals are zero. The double line represents the heavy meson D or D^*, while the dashed lines denote the pseudo-Goldstone bosons.

renormalization prescription determines the appropriate renormalization of couplings in the $\mathcal{O}(p^3)$ effective Lagrangian as discussed in section 2.5. Using two prescriptions makes possible to estimate the size of the counterterms, that are otherwise neglected. Further, we consider only contributions coming from the a_1^{NDR} part of the weak Lagrangian, as a_2^{NDR} is suppressed compared to a_1^{NDR} (**??**)(see also [**?**]).

Writing down the most general one loop graphs with two outgoing Goldstone bosons, K^0 and $\bar{K^0}$, one arrives at 26 Feynman diagrams. A number of these give zero contributions or are suppressed and are shown on Figures 5.1, 5.2, 5.3. The graphs that do contribute to the $D^0 \to K^0\bar{K^0}$ decay are shown on Fig. 5.4. Note that the factorizable loops, which renormalize vertices are omitted, as they contribute only at higher order in the chiral expansion (they do appear, however, in the loop determination of the α coupling related to f_D. For more details see section 2.5.)

To shorten the notation, the common factors in the S matrix have been factored out, so that the amplitude is written as

$$M(D^0 \to K^0\bar{K^0}) \; = \; -\frac{G_F}{\sqrt{2}}\, a_1^{\mathrm{NDR}}\, V_{us}\, V_{cs}^*\, \frac{F}{8\pi^2}\sqrt{m_D}, \tag{5.2}$$

where $F = \sum_n F_n$ is the sum of the amplitudes corresponding to the graphs on Fig. 5.4. In (5.2)

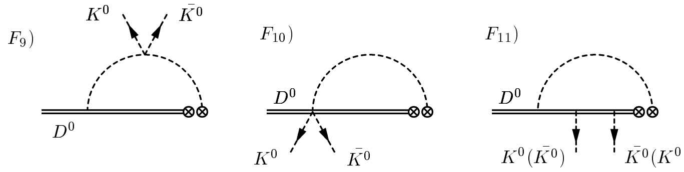


Figure 5.3: Power suppressed diagrams (neglected in the calculation).

we have also neglected the contributions of order $V_{ub}V_{cb}^*$, so that we use $V_{us}V_{cs}^* = -V_{ud}V_{cd}^*$. The partial decay width for the decay $D^0 \to K^0\bar{K}^0$ is then

$$\Gamma_{D^0\to K^0\bar{K}^0} = \frac{1}{2\pi}\frac{G_F^2}{8m_D}\left(a_1^{\rm NDR}\right)^2 |V_{us}\,V_{cs}^*|^2 \frac{|F|^2}{(8\pi^2)^2}|\vec{p}|, \tag{5.3}$$

where $\vec{p}$ is the K^0 three-momentum in the D^0 rest frame

$$|\vec{p}| = \frac{1}{2}\sqrt{m_D^2 - 4m_K^2}\ . \tag{5.4}$$

The nonzero amplitudes corresponding to the graphs on Fig. 5.4 are

$$F_1 + F_2 + F_3 = \frac{g\alpha}{f^2}\frac{13}{4}\left[\bar{B}_{00}(m_\pi,\Delta_d^*) - \bar{B}_{00}(m_K,\Delta_s^*)\right], \tag{5.5}$$

$$\begin{aligned} F_4 = -\frac{\alpha}{3f^2}\frac{m_D}{2}\Big\{&\left(m_D^2 - 2m_K^2\right)\left[B_0(m_D^2,m_K^2,m_K^2) - B_0(m_D^2,m_\pi^2,m_\pi^2)\right] + \\ &+ m_D^2\left[B_{11}(m_D^2,m_\pi^2,m_\pi^2) - B_{11}(m_D^2,m_K^2,m_K^2)\right] + \\ &+\left[B_{00}(m_D^2,m_\pi^2,m_\pi^2) - B_{00}(m_D^2,m_K^2,m_K^2)\right] + (m_\pi^2 - m_K^2)B_0(m_D^2,m_\pi^2,m_\pi^2)\Big\}, \end{aligned} \tag{5.6}$$

$$F_5 + F_6 = -\frac{\alpha m_D}{f^2}\frac{7}{24}\left[A_0(m_\pi^2) - A_0(m_K^2)\right], \tag{5.7}$$

$$\begin{aligned} F_7 + F_8 = -\frac{\alpha}{4f^2}\Big\{&\bar{B}_{00}(m_K,\tilde{\Delta}_d) + \bar{B}_{11}(m_K,\tilde{\Delta}_d) - \bar{B}_{00}(m_\pi,\tilde{\Delta}_s) - \bar{B}_{11}(m_\pi,\tilde{\Delta}_s) \\ &+ m_D\Delta_d\bar{B}_0(m_K,\tilde{\Delta}_d) - m_D\Delta_s\bar{B}_0(m_\pi,\tilde{\Delta}_s) + \frac{m_D}{2\tilde{\Delta}_d}A_0(m_K^2) - \frac{m_D}{2\tilde{\Delta}_s}A_0(m_\pi^2)\Big\}, \end{aligned} \tag{5.8}$$

where $\Delta_q^{(*)} = m_{D_q^{(*)}} - m_{D^0}$ and $\tilde{\Delta}_q = m_D/2 + \Delta_q$ for $q = d, s$. Note that $\tilde{\Delta}_q$ are of the order $m_D/2$, a consequence of relatively high momenta flowing in the loops of graphs F_7, F_8. The one and two point functions $A_0(m^2)$,$B_{0,00,11}(k^2,m^2,m^2)$, $\bar{B}_{0,00,11}(m,\Delta)$ appearing in the amplitudes (5.5)-(5.8) were defined in section 3.1. Explicit expressions can be found in section 3.4 and in appendix **??**.

It should be noted that in Eqs. (5.5)-(5.8) all the expressions vanish in the exact $SU(3)$ limit, where $m_K \to m_\pi$ and $\Delta_s \to \Delta_d$, $\tilde{\Delta}_s \to \tilde{\Delta}_d$. This shows explicitly, that the $D^0 \to K^0\bar{K}^0$ decay mode is a manifestation of the $SU(3)$ breaking effects (as already noted by H. Lipkin [**?**], if U symmetry is exact, then $\Gamma(D^0 \to K^0\bar{K}^0) = 0$).

The amplitudes shown on Figs. 5.1, 5.2, 5.3 are either exactly zero or are suppressed by powers of $1/m_D$ and g. The amplitudes corresponding to the diagrams on Figs. 5.1, 5.2 are zero due to symmetry reasons (because there are no such couplings in the heavy sector chiral Lagrangian (2.43), or because of Lorentz covariance), while the amplitudes F_9, F_{10} and F_{11} shown on Fig. 5.3 are power suppressed. An analysis of the loop integrals leads to the conclusion that $F_9 \sim g\big(\tilde{q}/m_D\big)^2 F_4$,

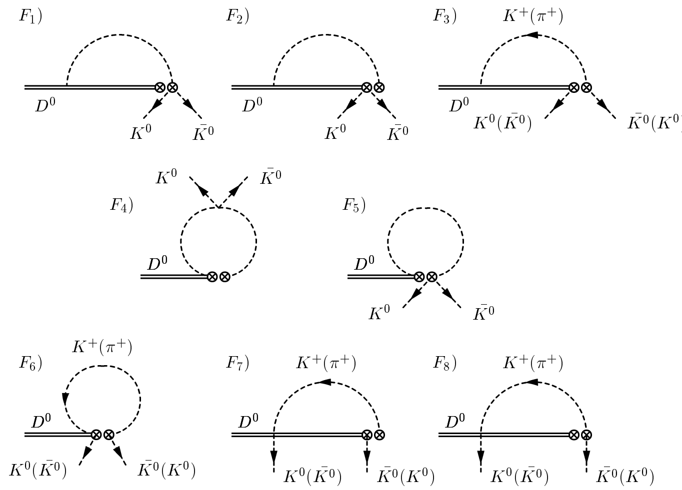


Figure 5.4: The nonzero diagrams in the $D^0 \to K^0 \bar{K^0}$ decay.

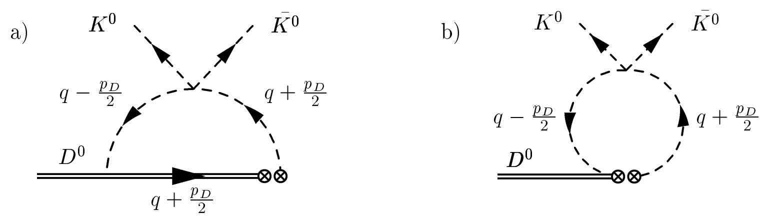


Figure 5.5: The momenta flowing in the graphs corresponding to a) the power suppressed F_9 amplitude and b) the leading contribution F_4 amplitude.

$F_{10} \sim g\big(\tilde{q}/m_D\big)F_4$ and $F_{11} \sim g^3\big(\tilde{q}/m_D\big)F_4$, where $\tilde{q}$ is a typical loop momentum less than $m_D/2$, so that the suppression need not be substantial. However, a direct evaluation of the amplitude F_{10} shows, that it is about 10 times smaller than F_4. Therefore, in our numerical calculation we neglect contributions of F_9, F_{10} and F_{11}. Numerical results are listed in Table **??**, section **??**.

5.2 The nonfactorizable color-current contributions

In this section we will estimate the contributions of $Q_{1,2}^{q(8)}$ operators in the weak Lagrangian (**??**). In the factorization limit the product of colored currents does not contribute at the meson level, as mesons are color singlet objects. At quark level, however, the colored currents can contribute through the gluon condensate. In order to estimate this contribution, we have to establish the connection between the underlying quark-gluon dynamics and the meson level picture. This is done through the use of the Heavy-Light Chiral Quark Model (HLχQM).

5.2.1 The Heavy-Light Chiral Quark Model

In the χQM [**?**, **?**, **?**, **?**, **?**, **?**], the light quarks (u, d, s) couple to the would-be Goldstone octet mesons (K, π, η) in a chiral invariant way. All effects are in principle calculable in terms of physical quantities and a few model dependent parameters, the quark condensate, the gluon condensate and the constituent quark mass [?, ?, ?]. Among many approaches the Chiral Quark Model (χQM)[?] was shown to be able to accommodate the intriguing $\Delta I = 1/2$ rule in the $K \to \pi\pi$ decays, as well as the CP violating parameters, by systematic involvement of the soft gluon emission forming gluon condensates and chiral loops at the order $\mathcal{O}(p^4)$ [?]. Also, in the "generalized factorization" it was shown [?], that the inclusion of gluon condensates is important in order to understand the $\Delta I = 1/2$ rule in the $K \to 2\pi$ decays.

As the χQM approach successfully indicated the main mechanisms in the $K \to \pi\pi$ decays, it seems worthwhile to investigate the decays of charm mesons within a similar framework. In the case of D meson decays one has to extend the ideas of the χQM to the sector involving a heavy quark (c) using the chiral symmetry of the light degrees of freedom as well as the heavy quark symmetry. This leads to the formulation of the Heavy-Light Chiral Quark Models (HLχQM) [?, ?, ?, ?, ?].

The Lagrangian of the HLχQM is

$$\mathcal{L} = \mathcal{L}_{\mathrm{HQ}} + \mathcal{L}_{\chi\mathrm{QM}} + \mathcal{L}_{\mathrm{Int}}, \tag{5.9}$$

where

$$\mathcal{L}_{\mathrm{HQ}} = \overline{Q}_v\, iv \cdot D\, Q_v + \mathcal{O}(m_Q^{-1}), \tag{5.10}$$

is the leading order Lagrangian of the heavy quark effective theory [?] (cf. Eq. (2.33)), with v^μ the heavy quark velocity and D_μ the covariant derivative containing gluon field. The chiral quark model Lagrangian $\mathcal{L}_{\chi\mathrm{QM}}$ is

In the heavy-light case, the generalization of the meson-quark interactions in the pure light sector χQM is given by the following $SU(3)_V$ invariant Lagrangian [?, ?, ?, ?]

$$\mathcal{L}_{\mathrm{Int}} = -G_H \left[\bar{\tilde{\Psi}}_a\, \overline{H}_{va}\, Q_v \,+\, \overline{Q}_v\, H_{va}\, \tilde{\Psi}_a \right], \tag{5.11}$$

with H_{va} the heavy meson field (2.38). The dependence on heavy-quark velocity v is denoted explicitly, while a is the flavor index. The unknown constant G_H can be related to constants α, g of HHχPT as described below (cf. Eqs. (5.17)-(??)).

The weak currents have the usual form, except that the Goldstone bosons are factored out. The weak current with two light quarks is

$$\bar{q}_L \gamma^\mu \lambda^a q_L \;=\; \bar{\tilde{\Psi}}_L \gamma^\mu \Lambda^a\, \tilde{\Psi}_L, \qquad \Lambda^a \;\equiv\; \xi^\dagger \lambda^a\, \xi. \tag{5.12}$$

The weak current with one heavy quark is as given by HQET [?], except that as before, the Goldstone bosons are factored out

$$J_c^\mu = C_\gamma(\mu) \bar{\tilde{\Psi}}_b \xi^\dagger_{bc} \gamma^\mu P_L Q_v \,+\, C_v(\mu) \bar{\tilde{\Psi}}_b \xi^\dagger_{bc} v^\mu P_L Q_v, \tag{5.13}$$

The coefficients $C_{\gamma,v}$ are determined from the QCD renormalization for $\mu < m_c$. However, for $\mu \simeq \Lambda_\chi$, $C_\gamma \simeq 1$ and $C_v \simeq 0$. When quark fields will be integrated out, this will lead to the leading order term of the current (2.46).

5.2.2 Estimate of the color-current contributions

We are now able to outline the strategy used in [?] to estimate the contribution of the nonfactorizable colored currents to $D^0 \to K^0 \bar{K}^0$. We will not discuss all the details, for which we refer the

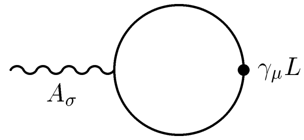


Figure 5.6: Feynman diagram for the bosonization of the left-handed current to the order $\mathcal{O}(p)$.

reader to [?, ?]. The key observation is, that once the quark degrees of freedom are integrated out, one has to end up with the most general effective Lagrangian containing the meson fields, i.e., the HHχPT Lagrangian (5.9). By integrating out the quark fields one can thus (i) connect the unknown couplings m_χ, G_H in (5.9) to the constants of the HHχPT, that are fixed from the experiment, (ii) calculate (in a model dependent way) the higher order constants of the HHχPT, (iii) relate the constants of the HHχPT to each other. For instance, the lowest order chiral Lagrangian in the light pseudoscalar sector (2.28) can be obtained by coupling two axial fields to a quark loop using the Lagrangian in Eq. (**??**):

$$i\mathcal{L}^{(2)}_{\text{str}} = -N_c \int \frac{d^dp}{(2\pi)^d} \operatorname{Tr}\left[(\gamma_\sigma\gamma_5\mathcal{A}^\sigma)\; S(p)\; (\gamma_\rho\gamma_5\mathcal{A}^\rho)\; S(p)\right] \sim \operatorname{Tr}\left[\mathcal{A}_\mu\mathcal{A}^\mu\right], \tag{5.14}$$

where $S(p) = (\not p - m_\chi)^{-1}$, and the trace is both in flavor and Dirac spaces. The result on the right-hand side of Eq. (5.14) is the standard form of the lowest order chiral Lagrangian (2.28), as can easily be seen by using the relations

$$\mathcal{A}_\mu = -\frac{1}{2i}\xi\,(\partial_\mu\Sigma^\dagger)\,\xi = \frac{1}{2i}\xi^\dagger\,(\partial_\mu\Sigma)\,\xi^\dagger. \tag{5.15}$$

Similarly one obtains the lowest order $\mathcal{O}(p)$ strong chiral Lagrangian (2.43) in the heavy sector.

In a similar way, i.e., by integrating out the quark fields, we can dress-up the quark weak currents with mesonic fields. This has been called the process of bosonization in [?]. Let us consider the bosonization of the light weak current. The lowest order term $\mathcal{O}(p)$ is obtained, when the vertex Λ^a from (5.12) and the axial vertex ($\sim \mathcal{A}_\mu$) from (**??**) are combined with the quark loops (see Fig. 5.6):

$$j^a_\mu(\mathcal{A}) = -\,iN_c \int \frac{d^dp}{(2\pi)^d} \operatorname{Tr}\left[(\gamma_\mu L\,\Lambda^a)\,S(p)\;(\gamma_\sigma\gamma_5\mathcal{A}^\sigma)\;S(p)\right] \;\sim\; \operatorname{Tr}\left[\Lambda^a\,\mathcal{A}_\mu\right]. \tag{5.16}$$

This coincides with (2.30) when (5.15) is used.

Note that the proportionality factors in (5.14), (5.16) contain divergent integrals. These can be regulated in different ways, but in this context they are treated as free parameters. They are used to relate different model parameters and chiral Lagrangian constants to each other. For instance to get a leading order estimate of coupling G_H, one uses the self-energy diagrams of heavy mesons and light-pseudoscalars. A logarithmically divergent integral is contained in both calculations and is used to relate f_π to G_H

$$G_H \simeq \frac{2\sqrt{m_\chi}}{f_\pi}. \tag{5.17}$$